Solving Polynomial Eigenproblems by Linearization

Nicholas J. Higham
School of Mathematics, The University of Manchester
Sackville Street, Manchester, M60 1QD, UK
higham@ma.man.ac.uk
http://www.ma.man.ac.uk/~higham/

March 3, 2005

A matrix polynomial (or λ-matrix) has the form

$$P(\lambda) = \lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + A_0,$$

where $A_k \in \mathbb{C}^{n\times n}$, $k = 0\colon m$. Associated with $P(\lambda)$ is the polynomial eigenvalue problem (PEP) of finding eigenvalues λ and corresponding right eigenvectors $x \neq 0$ and left eigenvectors $y \neq 0$ satisfying $P(\lambda)x = 0$ and $y^*P(\lambda) = 0$. The polynomials with $m > 1$ of greatest practical importance are the quadratics, and the corresponding eigenvalue problem $Q(\lambda)x = (\lambda^2 A + \lambda B + C)x = 0$ is the quadratic eigenvalue problem (QEP). QEPs arise in various applications, including vibration analysis of structural systems acoustic structural coupled systems, fluid mechanics, MIMO systems in control theory, and signal processing [4].

A standard way of solving the PEP is to convert the matrix polynomial $P(\lambda)$ into a linear polynomial

$$L(\lambda) = \lambda X + Y, \qquad X, Y \in \mathbb{C}^{mn\times mn}$$

and to solve the resulting generalized eigenproblem $L(\lambda)x = 0$ by the QZ algorithm, for small to medium size problems, or a Krylov algorithm for large sparse problems. Linearization increases the size of the problem, doubling it in the case of the QEP. We are interested in pencils L that are linearizations of P in the following sense: they satisfy

$$E(\lambda)L(\lambda)F(\lambda) = \begin{bmatrix} P(\lambda) & 0 \\ 0 & I_{(m-1)n} \end{bmatrix}$$

for some unimodular $E(\lambda)$ and $F(\lambda)$ (that is, $\det(E(\lambda))$ must be a nonzero constant, independent of λ, and likewise for F). This definition implies that $c\det(L(\lambda)) = \det(P(\lambda))$ for some nonzero constant c, so that L and P have the same spectrum.

The importance of condition numbers for characterizing the sensitivity of solutions to problems is widely appreciated. Condition estimation algorithms for standard and generalized eigenproblems are now used in most of the major mathematical program libraries as well as some commercial scientific software. However, practically oriented analysis of condition numbers of the PEP for $m \geq 2$ is not yet as well developed and understood.

A normwise condition number of a simple, finite, nonzero eigenvalue λ of a regular PEP with corresponding right eigenvector x can be defined by

$$\kappa_P(\lambda) = \lim_{\epsilon \to 0} \sup \left\{ \frac{|\Delta\lambda|}{\epsilon|\lambda|} : (P(\lambda+\Delta\lambda)+\Delta P(\lambda+\Delta\lambda))(x+\Delta x) = 0,\ \|\Delta A_k\|_2 \le \epsilon\omega_k,\ k = 0\colon m \right\}.$$

The ω_k are nonnegative weights that allow flexibility in how the perturbations are measured; in particular, ΔA_k can be forced to be zero by setting $\omega_k = 0$. An explicit formula for this condition number is given by [3, Thm. 5]

$$\kappa_P(\lambda) = \frac{(\sum_{k=0}^{m} |\lambda|^k\ \omega_k)\|y\|_2\|x\|_2}{|\lambda|\ |y^* P'(\lambda)x|}. \tag{1}$$

Important questions include the following:

- How should we choose a linearization from among the infinitely many available? Criteria to apply include conditioning and preservation of structure.
- For a given pencil, under what conditions is it a linearization of a given polynomial P?
- How does the choice of linearization affect the performance of numerical methods?

In the talk we will mainly concentrate on the first question, concentrating on conditioning effects. We now outline recent work in this direction.

Two important sets of potential linearizations are identified by Mackey, Mackey, Mehl and Mehrmann [2]. With the notation

$$\Lambda = [\lambda^{m-1}, \lambda^{m-2}, \ldots, 1]^T,$$

the sets are

$$\mathbb{L}_1(P) = \{\, L(\lambda) : L(\lambda)(\Lambda \otimes I_n) = v \otimes P(\lambda),\ v \in \mathbb{C}^m \,\}, \tag{2}$$
$$\mathbb{L}_2(P) = \{\, L(\lambda) : (\Lambda^T \otimes I_n)L(\lambda) = w^T \otimes P(\lambda),\ w \in \mathbb{C}^m \,\}. \tag{3}$$

Here, $\otimes$ denotes the Kronecker product, given by $A \otimes B = (a_{ij}B)$. It is proved in [2] that $\mathbb{L}_1$ and $\mathbb{L}_2$ are vector spaces and that almost all pencils in these spaces are linearizations of $P(\lambda)$. The underlying reason for the interest in $\mathbb{L}_1$ and $\mathbb{L}_2$ is that the right eigenvectors of P can be recovered from the right eigenvectors of pencils in $\mathbb{L}_1$, while the left eigenvectors of P can be recovered from the left eigenvectors of pencils in $\mathbb{L}_2$. It is natural to concentrate attention on the pencils that lie in

$$\mathbb{DL}(P) = \mathbb{L}_1(P) \cap \mathbb{L}_2(P),$$

because there is a simultaneous correspondence between left *and* right eigenvectors of P and of pencils in $\mathbb{DL}(P)$. It is shown in [2] that for any $L \in \mathbb{DL}(P)$ we have $w = v$ in (2) and (3) and that $\mathbb{DL}(P)$ is always an m-dimensional vector space of pencils associated with P. In this talk we focus on linearizations in $\mathbb{DL}(P)$.

Using our results on eigenvalue conditioning for PEPs and generalized eigenproblems, we obtain an expression for the condition number of a simple, finite or infinite, eigenvalue of a pencil $L(\lambda) = \lambda X + Y \in \mathbb{DL}(P)$ in terms of P and v, with minimal explicit dependence on X and Y. The linearizations with $v = e_1$ and $v = e_m$ are shown always to be *almost optimal within* $\mathbb{DL}(P)$ for eigenvalues of modulus greater than or less than 1, respectively, provided that the measure $\rho = \max_i \|A_i\|_2 / \min(\|A_0\|_2, \|A_m\|_2)$ of the scaling of the problem is of order 1. Under the same scaling assumption these two linearizations are shown to be *about as well conditioned as the original polynomial* P. Thus, despite the structured nature of these two pencils L (including zero blocks and repeated occurrences of the A_k), arbitrary perturbations to L do no more damage to λ than perturbations to the A_k in the original PEP.

For quadratic polynomials that are "not too heavily damped", and in particular for elliptic quadratics, a simple scaling is shown to convert the problem to one for which $\rho \approx 1$.

We also analyze the eigenvalue conditioning of the widely used first and second companion linearizations. Their conditioning relative to that of the original polynomial is shown to depend crucially on the left eigenvectors of the linearization and of P, and the conditioning of the companion forms is guaranteed to be similar to that of P if the 2-norms of the coefficient matrices are all approximately 1.

The significance of these results is twofold. First, they provide theoretical support for the widely used approach of solving PEPs by linearization. Second, they show how to choose an almost optimally conditioned linearization. For full details of these results, and numerical experiments, see [1].

References

[1] Nicholas J. Higham, D. Steven Mackey, and Françoise Tisseur. The conditioning of linearizations of matrix polynomials. Numerical analysis report, Manchester Centre for Computational Mathematics, Manchester, England, 2005. In preparation.

[2] D. Steven Mackey, Niloufer Mackey, Christian Mehl, and Volker Mehrmann. Vector spaces of linearizations for matrix polynomials. Numerical analysis report, Manchester Centre for Computational Mathematics, Manchester, England, 2005. In preparation.

[3] Françoise Tisseur. Backward error and condition of polynomial eigenvalue problems. *Linear Algebra Appl.*, 309:339–361, 2000.

[4] Françoise Tisseur and Karl Meerbergen. The quadratic eigenvalue problem. *SIAM Rev.*, 43(2):235–286, 2001.