Configurational forces in optimisation algorithms

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1. Introduction

One of the key ingredients of Newtonian mechanics is the duality between displacements and forces. Here, displacements are understood as a change of position in space of a certain material particle. It is less straightforward to define a force [1], but forces are understood to be the conjugated variables to the displacements. That is one can obtain the potential (or complementary) energy of a system by integrating the forces with respect to the displacements (or vice versa). Equivalently, forces can be obtained by differentiating the energy with respect to the displacements.
The above viewpoint is that of Lagrange, i.e. a material particle is followed during its motion in space. The opposite viewpoint is due to Euler, whereby a fixed position in space is taken and material flow through this point is considered. This is an alternative way of describing the same motion, and the involved displacements are of a different nature than the displacements that are normally considered in solid mechanics. A test that involves both types of displacements is crack propagation due to mode I loading. Here, an example of the former type of displacements is the particle motion *parallel* to the loading direction. Conversely, the extension of the crack tip *perpendicular* to the loading direction constitutes an example of the latter type of displacements.

2. Configurational forces

Two distinct types of displacements imply the existence of two distinct types of conjugated forces. *Physical forces* (a.k.a. spatial forces) are the forces as they are commonly known, i.e. those forces that are conjugated to displacement of material through space, whereas *configurational forces* (or material forces) are conjugated to displacement of a spatial configuration through the material[1]. The two descriptions of mechanics are treated in detail in [1] and [2], with emphasis on continuum mechanics and engineering mechanics, respectively.
An unbalance of physical forces denotes a lack of equilibrium. On the other hand, an unbalance of configurational forces denotes inhomogeneity in the material, as has been amply demonstrated in [1–3], among others. As such, cracks, interfaces, inclusions, dislocations and disclinations all lead to the occurence of non-zero configurational forces. For instance, the J-integral of a crack tip and the Peach-Koehler force on a dislocation are in fact configurational forces. But apart from these physical causes, non-zero configurational forces may also occur as the result of numerical discretisation. As argued in [4], a sub-optimal finite element discretisation leads to non-smooth stress fields, which in turn result in a configurational force residual. In fact, there is no difference between stress jumps resulting from a material inhomogeneity and stress jumps resulting from the discretisation.

[1]The chosen nomenclature is a matter of personal preference. “Spatial and material forces” emphasises the duality that exists between them and relates to terminology of finite strain applications. “Physical and configurational forces” perhaps better reflects the underlying driving mechanisms.

3. Optimisation based on configurational force equilibrium

This had led to the idea that configurational forces can be used to optimise finite element discretisations, see for instance the works of Mueller and coworkers [4–6]. The underlying principle is that (displacement-based) finite element methods overestimate the potential energy of the system, and lower values of the potential energy can be found by shifting the nodal positions. Specifically, an additional release of energy is obtained by moving the nodes *opposite* to the configurational force residual, and the optimal nodal coordinates are found when the configurational force residual vanishes. Whereas the earlier mesh optimisation strategies used ad-hoc algorithms to arrive at vanishing configurational forces, more recently a sound variational basis was formulated [7–9]. In particular, the potential energy of the system is minimised simultaneously with respect to the nodal displacements and the nodal coordinates[2]. The resulting system of equations can then be solved in an uncoupled [7, 9] or coupled [9] manner.

The same coupled equations can also be used to optimise truss lay-outs in structural mechanics [10]. The global system of equations takes a format similar to that of mesh optimisation: the degrees of freedom are again the nodal positions before and after the load is applied, the only difference being the specific form of the element stiffness matrices and residual vectors. Again the resulting configuration leads to the minimum potential energy that can be obtained for the given element connectivity.

4. Illustration: tendon lay-out optimisation in pre-stressed concrete

Another application of configurational force equilibrium in optimisation concerns the tendon lay-out in pre-stressed concrete. As this is merely an illustration of the above theory, a number of simplifications will be made such that a closed-form solution in basic engineering mechanics terms can be obtained:

- Euler-Bernoulli beam theory will be used. Moreover, only the axial normal stress (denoted σ) will assumed to be non-zero.
- Statically determinate beams will be considered, so that the stress distribution is known a priori for any given applied load. Consequently, the complementary energy (rather than the potential energy) of the system will be computed.
- The cross section of the beam will be taken rectangular with width b and height h. The length of the beam will be denoted ℓ.
- Purely horizontal tendons will be assumed at a position Z from the centroid of the beam.
- No bond between steel and concrete is taken into account. The pre-stress force P does not depend on the axial coordinate x.
- The area A and the second moment of inertia I are supposed to be unaffected by the tendon.

The stress is written as

$$\sigma = -\frac{P}{A} + \frac{(M - PZ)\,z}{I} \qquad \text{where} \quad -\tfrac{1}{2}h \leq z \leq \tfrac{1}{2}h \tag{1}$$

where M is the bending moment due to any applied loads (i.e. the excentricity of the prestress load is not included in M). The complementary energy $\mathcal{U}$ of the system can then be written as

$$\mathcal{U} = \int_{\Omega} \frac{1}{2E}\,\sigma^2 \mathrm{d}V \tag{2}$$

where E is the Young's modulus of the material.

[2] In the terminology of finite strain theory: the energy is minimised with respect to the spatial as well as the material coordinates of the nodes.

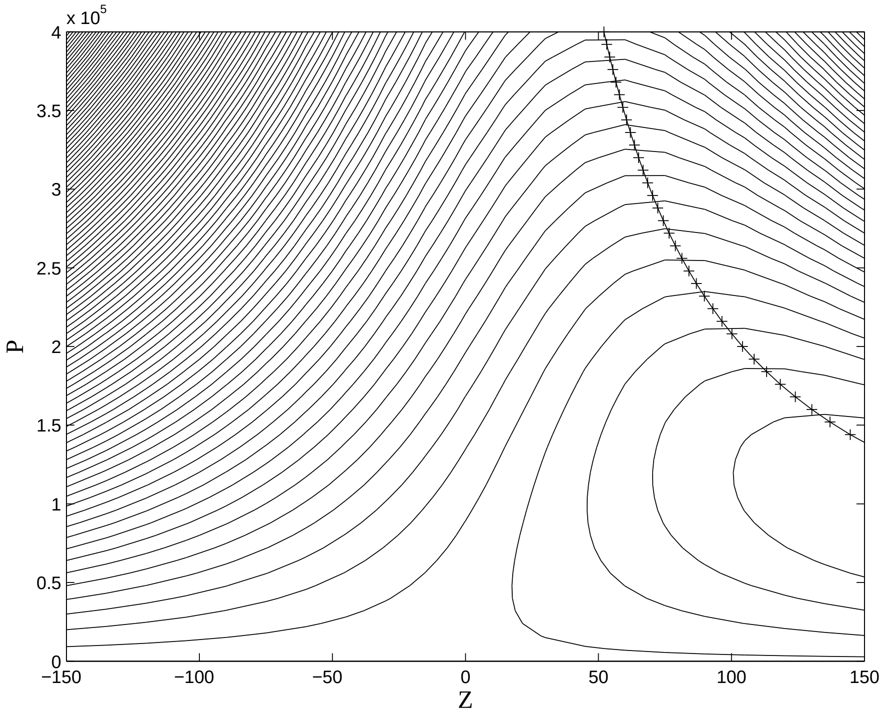

FIGURE 1. Contour plot of complementary energy and optimal tendon positions for given values of P.

A configurational force residual f_Z can be identified that is energy conjugated to the tendon position Z:

$$f_Z = \frac{\partial \mathcal{U}}{\partial Z} = \frac{1}{E} \int_\Omega \sigma \frac{\partial \sigma}{\partial Z} \,\mathrm{d}V = \frac{1}{E} \int_\Omega \left(-\frac{P}{A} + \frac{(M - PZ)\, z}{I} \right) \frac{-Pz}{I} \,\mathrm{d}V \tag{3}$$

Expanding the volume integral gives

$$f_Z = \frac{b}{E} \int_0^\ell \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \left(\frac{P^2 z}{AI} + \frac{-PMz^2 + P^2 Z z^2}{I^2} \right) \mathrm{d}z\mathrm{d}x = -\frac{P}{EI} \int_0^\ell (M - PZ)\,\mathrm{d}x \tag{4}$$

The last integal in expression (4) is also equal to the *average* value of the bending moment times the length of the beam ℓ. Therefore,

$$f_Z = -\frac{P}{EI} \int_0^\ell (M - PZ)\,\mathrm{d}x = -\frac{P\ell}{EI} \left(M^{\mathrm{aver}} - PZ \right) \tag{5}$$

where M^{aver} is the average value of M over the beam length[3]. Finding the optimal tendon position is equivalent to requiring the configurational force residual to vanish, that is

$$f_Z = 0 \quad \Longrightarrow \quad Z = M^{\mathrm{aver}}/P \tag{6}$$

In Figure 1 a contour plot is given of the complementary energy following from Equation (2) as a function of the prestress load P and the tendon position Z. The following input parameters were used: $E = 30000$ MPa, $\ell = 5$ m, $h = 300$ mm and $b = 100$ mm. Moreover, a uniformly distibuted load $q = 10$ kN/m was assumed. The beam was simply supported, therefore the average bending moment due to the uniformly distributed load equals $M^{\mathrm{aver}} = \frac{1}{12} q \ell^2$. The optimal position Z of the tendon follows then from Equation (6) for given values of the prestress force P. This curve (P versus optimal Z) has also been plotted in Figure 1. It can

[3]Defining an average curvature $\kappa^{\mathrm{aver}} \equiv (M^{\mathrm{aver}} - PZ)/EI$, the configurational force residual can also be written as $f_Z = -P\ell\, \kappa^{\mathrm{aver}}$, which emphasises the relation between the deformation and the configurational force.

be verified that this curve indeed indicates the mimimum values of complementary energy for any given value of P

5. Closure

As has been argued, configurational forces can be used in optimisation algorithms. Configurational forces are an indication for inhomogeneity, and requiring that the configurational force residual vanishes is equivalent to requiring the conjugated displacement variable to attain its optimal value. These conjugated displacements are normally quantities that define the geometry of the problem, rather than the positions of the material particles after the load is applied. Earlier examples of optimisation by using configurational forces include mesh optimisation (i.e. optimising the nodal positions) and truss optimisation (i.e. optimising the joint positions).

In this paper, a third type of optimisation is treated in more detail, namely optimisation of the tendon position in reinforced concrete. For a simplified problem statement, a closed-form expression has been found that relates the tendon position to the average value of the bending moment and the value of the prestress force. Although the presented example has been kept at its simplest, the method offers potential for further applications of tendon layout optimisation. Issues that need addressing include:

- The extension to statically indeterminate structures needs to be made. This implies that a simultaneous equilibration of physical forces and configurational forces is needed, similar to the coupled method formulated and implemented in [9, 10].
- All stress components should be included in the energy functional.
- More sophisticated tendon models should be adopted, including curved tendons and bond between tendon and concrete.
- In the present formulation, no distinction is made between volumes with tensile stress and volumes with compressive stress. Relevance to engineering practice requires that the optimisation favours volumes with compressive stresses over volumes with tensile stresses.

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