Integrated thickness and shape optimization of linear shells

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1. Introduction

Structural optimization has drawn the attention to many researchers for a few decades. Mathematical programming methods used in structural optimization are generally based on two categories. First method is based on gradient methods which normally use objective function, derivatives of objective function and constraint functions. The other method based on non-gradient methods which use only objective function and constraint functions. A gradient-based method is expected to be more efficient since it requires more information in each analysis. Therefore it needs fewer iterations of structural analysis. Researches in structural optimization may be categorized into three main areas namely shape optimization, thickness optimization and topology optimization.

The first research on optimisation of structures was Zienkiewicz and Campbell [1]. They have discussed the problem of finding the optimum shape of two-dimensional structures known as shape optimisation. Since then several researchers have contributed in this area such as Choi and Haug[2] and Haftka and Grandhi[3]. General methods to obtain the optimum structure is varying the shape of an initial structure. Therefore solutions obtained from shape optimisation methods maintain the same thickness of structures. In sizing optimisation problems, design variables are cross-sectional areas (beams and trusses) or the thicknesses (plate and shell structures) while the geometric shape of structures remains unchanged. Optimality criterion are generally employed for this area[4].

The objective of the present work is to incorporate fully automated mesh generation scheme and analytical sensitivity analysis into integrated shape and thickness optimisation of shell structures. First the fully automated mesh generation scheme for the model generation is discussed. Three main objectives in the model generation phase are to implement the design element concept, to choose the design variables which are shapes and thicknesses of structures and to smooth the geometry of the structures. Next the proposed analytical design sensitivities are discussed. These sensitivities are obtained from differentiation of the finite element equations with respect to design variables. Finally, examples validating the technique are presented and compared with exact solutions and others' works.

2. Sensitivity Analysis

The major difficulty in shape optimization problem is that the shape of structures change in every iteration during the optimization process. This means that using the same finite element mesh may cause distortion which may lead to inaccurate results from finite element analysis. Therefore structures need to be re-meshed each iteration of optimization process. In order to save computational time, automatic mesh generation scheme[5] using the design element concept and isoparametric technique is employed.

2.1 Automatic Mesh Generation

Automatic mesh generation scheme used in this paper employs the isoparametric technique. In this method, shell structures are divided into design elements as shown in figure 1a. These design elements consist of master nodes that define shape of the mid-surface and thickness of each master node. The shape of design elements are defined by nine master nodes. Therefore edges of design elements are represented by parabolic curves. Each design element is then divided into finite element mesh preserving geometric regularity and avoiding distortion. A finite element model obtained from mesh generation scheme is depicted in figure 1b. Coordinates of any point j at the mid-surface shells can be written as

$$x_{mj} = \sum_{i=1}^{9} f_i(r_j, s_j)X_i, \quad y_{mj} = \sum_{i=1}^{9} f_i(r_j, s_j)Y_i, \quad z_{mj} = \sum_{i=1}^{9} f_i(r_j, s_j)Z_i \tag{1}$$

Where x_{mj}, y_{mj}, and z_{mj} are coordinates of point j at the mid-surface. X_i, Y_i, and Z_i are coordinates of master node i. $f_i(r, s)$ is the i^{th} shape function of the Lagrangian nine node element. Eighteen-node hexahedral shell finite element, nine nodes at top and bottom surfaces displayed in figure 2, is employed to perform analysis of shell structure. Therefore coordinates of top and bottom surfaces of finite element models are needed and can be defined from coordinates and thicknesses of master nodes as

$$x_j = \sum_{i=1}^{9} f_i(r_j, s_j)X_i + 1/2\zeta_j\Big(\sum_{i=1}^{9} f_i(r_j, s_j)t_i\Big)\hat{T}_x(r_j, s_j) \tag{2}$$

Likewise y_j and z_j can be written in the same form as x_j. Where x_j, y_j, and z_j are coordinates of point j of any point in hexahedral shell elements. $t_i, \hat{T}_x, \hat{T}_y$, and $\hat{T}_z$ are thicknesses of master node i and projection of unit vector of thickness directions on X, Y, and Z axis, respectively.

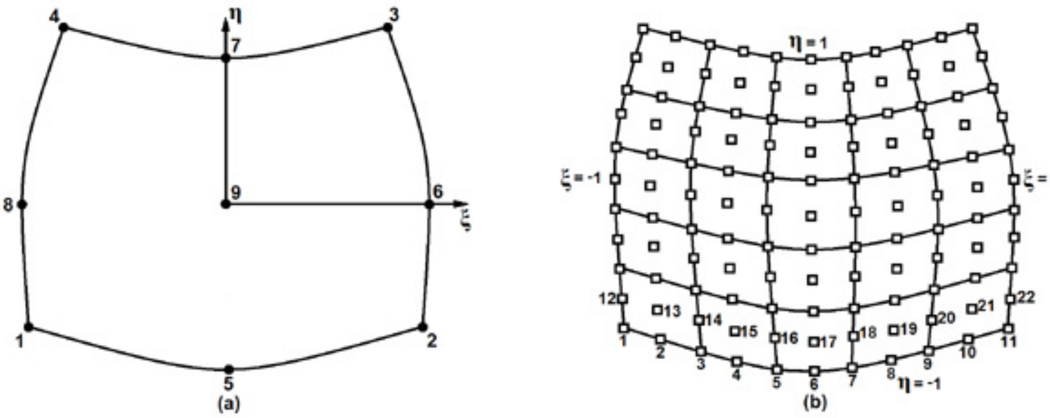

FIGURE 1. (a) Design element model. (b) Finite element model

2.2 Sensitivity Analysis using Design Elements Concept

Analytical design sensitivities are derived assuming the design elements concept at the equilibrium configuration. Equilibrium equation of shell finite element method is

$$[K]\{U\} = \{R\} \tag{3}$$

Where $[K], \{U\}$, and $\{R\}$ are structural stiffness matrix, displacement and applied load vectors, respectively. Differentiate stiffness matrix with respect to design variable, D_k, gives

$$[K]\left\{\frac{\partial U}{\partial D_k}\right\} + \left[\frac{\partial K}{\partial D_k}\right]\{U\} = \left\{\frac{\partial R}{\partial D_k}\right\} \tag{4}$$

Let $\hat{R}$ be $\left\{\frac{\partial R}{\partial D_k}\right\} - \left[\frac{\partial K}{\partial D_k}\right]\{U\}$, therefore equation (4) may be rewritten as

$$[K]\left\{\frac{\partial U}{\partial D_k}\right\} = \{\hat{R}\} \tag{5}$$

Where R is defined as pseudo load vector. It can be seen from equation (4) and (5) that to get derivative of U with respect to D_k, the derivative of stiffness matrix and load vector with respect to design variables have to be evaluated.

The derivative of a structural stiffness matrix can be obtained by assembling the derivatives of element stiffness matrices. Element stiffness matrix can be written as

$$[k]_e = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} [T]^T[B]^T[D][B][T]|J| \,\mathrm{d}\xi \,\mathrm{d}\eta \,\mathrm{d}\zeta \tag{6}$$

Where $[T]$, $[B]$, $[D]$ and $|J|$ are Transformation, strain-displacement, constitutive and determinant of Jacobian matrices, respectively. It is obvious that constitutive matrix does not depend on design variables therefore derivative of element stiffness matrix is

$$\begin{aligned}\left[\frac{\partial K}{\partial d_k}\right]_e &= \Bigg\{\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \frac{\partial [T]}{\partial d_k}^{T} [B]^T[D][B][T]|J| \,\mathrm{d}\xi + [T]^T \frac{\partial [B]}{\partial d_k}^{T} [D][B][T]|J| \\ &+ [T]^T[B]^T[D]\frac{\partial [B]}{\partial d_k}[T]|J| + [T]^T[B]^T[D][B][T]\frac{\partial |J|}{\partial d_k} \\ &+ [T]^T[B]^T[D][B]\frac{\partial [T]}{\partial d_k}|J|\Bigg\} \,\mathrm{d}\xi \,\mathrm{d}\eta \,\mathrm{d}\zeta \end{aligned} \tag{7}$$

Integrand of equation (7) is performed by numerical integration with 3x3x2 Gauss integration points.

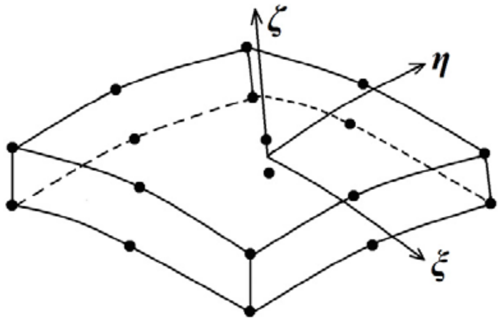

FIGURE 2. eighteen-node hexahedral element geometry

3. OPTIMISATION

Generally, combined shape and thickness optimisation problems can be mathematically stated as

$$\begin{aligned}&\text{Minimise} \quad F(d) \\ &\text{Subject to} \quad g_i(d) \le 0 \quad i = 1, 2, ..., n \\ &d_j^l \le d_j \le d_j^u \quad j = 1, 2, ..., m,\end{aligned} \tag{8}$$

where $F(d)$ and $g_i(d)$ are objective function and constraint functions, respectively. $\mathbf{d}$ is the vector of design variables. d_j^l and d_j^u are lower and upper bounds on design variables, respectively. n is number of constraints and m is number of design variables. Generally, three types of objective functions for structural problems are displacements, stresses and masses of structures.

4. EXAMPLES

4.1 Simply supported shell subjected to concentrated load.

Shallow shell subjected to concentrated force at the central, illustrated in figure 3, is investigated. Shell consists of 9 design nodes and is divided into 8x8 meshes. According to symmetry, only one quarter of structure is analysed. Young's modulus (E) and Poisson's ratio (υ) are 210.0x10^9 psi and 0.3, respectively. This example is to minimise the total strain energy. The design variables are z-coordinates and thicknesses of design nodes. The optimum design, depicted in figure 4a, will be compared with those from Ansola *et.al.*[8], illustrated in figure

4b. It can be seen from figure 4 that the optimum design shows similar result obtained from Ansola[8]. Moreover the result obtained is not beyond expectation since the optimum way to support applied load with the four simply support at corner is four bar truss-like structure which transfer the load to supports under compression.

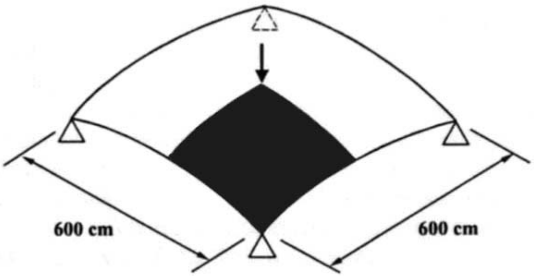

Figure 3. Geometry of simply supported shell subjected to central load

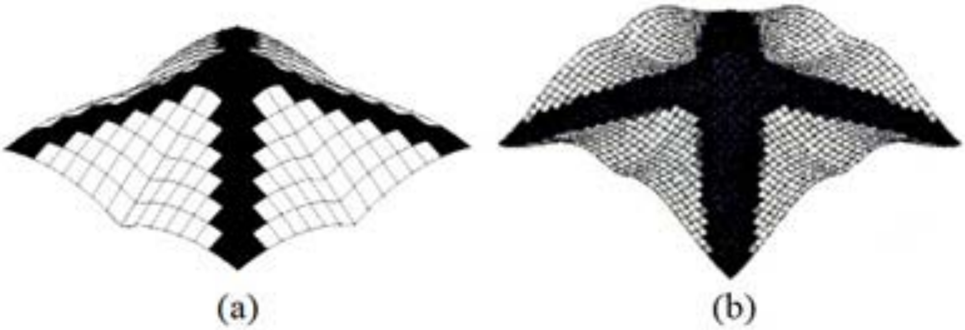

Figure 4. Optimum design of simply supported shell (a) proposed method (b) Ansola *et.al.*[8]

5. Conclusion

Integrated thickness and shape optimization of linear shells have been evaluated by using gradient-based method which can be applied to various shell structures. Proposed optimisation technique has been extensively assessed against geometrically linear benchmarks and perform well compared with exact solutions and others' works. Further work will be to extend the scope of study to nonlinear shell structures.

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