

# Dynamic refinement in smoothed particle hydrodynamics with applications in fluid flow problems.

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## Abstract

Smoothed Particle Hydrodynamics (SPH)[1] is a simple and effective numerical method for solving problems in computational mechanics. SPH is a fully Lagrangian meshless method [2] ideal for solving large deformation problems such as complex free surface fluid flows [3].

This paper presents a general method for dynamic particle refinement through particle splitting. A candidate particle is split into several ‘daughter’ particles according to a given refinement pattern centered about the original particle. Through the solution of a non-linear minimization problem the optimal mass distribution of the daughter particles is obtained so as to reduce the errors introduced to the underlying density field. This procedure necessarily conserves the mass of the system. Conservation of energy and momentum properties are also discussed.

This theory is applied to a simple fluid flow in 2D.

## 1. SPH For Fluid Flows

Smoothed Particle Hydrodynamics (SPH) and other allied meshless methods are based on the *Reproducing Kernel Approximation* of an arbitrary function  $f(\mathbf{x})$  given by,

$$\langle f(\mathbf{x}) \rangle = \int_D f(\mathbf{x}') w(\mathbf{x} - \mathbf{x}', h) d\mathbf{x}' \quad (1)$$

where  $w(\mathbf{x}, h)$  is a *Kernel Function* with compact support of radius  $2h$ .  $h$  is commonly referred to as the *Smoothing length* of the kernel. As  $w(\mathbf{x}, h)$  approaches the Dirac delta function, this integral approximation converges to the exact function  $\langle f(\mathbf{x}) \rangle \rightarrow f(\mathbf{x})$ .

By careful selection of kernel function it is possible to ensure that *any* polynomial of order less than  $k$  is exactly reproduced. In this case the approximation is said to be  $k^{\text{th}}$  *Order Consistent* and  $\langle p_j(\mathbf{x}) \rangle = p_j(\mathbf{x})$  for  $0 \leq j \leq k$ .

In order to develop a practical numerical scheme the integrals in Equation(1) are discretized to give,

$$f_h(\mathbf{x}) = \sum_{b \in M_{\mathbf{x}}} V_b f(\mathbf{x}_b) w_b(\mathbf{x}, h_b) \approx \langle f(\mathbf{x}) \rangle \quad (2)$$

where  $V_b$  is a *Volume* associated to the point  $b$  and  $w_b(\mathbf{x}, h_b) := w(\mathbf{x} - \mathbf{x}_b, h_b)$ . Since the kernel has a compact support the above summation is over the finite set of points  $M_{\mathbf{x}}$  that contribute to the approximation of  $f$  at the point  $\mathbf{x}$ .

The internal pressure forces for a fluid particle [5] with variable smoothing length are given by,

$$\mathbf{T}_a^P = \sum_b m_a m_b \left( \frac{P_b}{\rho_b^2} \nabla w_b(\mathbf{x}_a, h_b) - \frac{P_a}{\rho_a^2} \nabla w_a(\mathbf{x}_b, h_a) \right) \quad (3)$$

and the viscous forces are given by,

$$\mathbf{T}_a^{visc} = \sum_b V_a V_b (\boldsymbol{\sigma}'_a + \boldsymbol{\sigma}'_b) \nabla w_a(\mathbf{x}_b, h_a), \quad \boldsymbol{\sigma}' = 2\mu \mathbf{d}' \quad (4)$$

where  $\mathbf{d}'$  is the deformation gradient tensor  $\mathbf{d}' = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T) - \frac{1}{3}\text{tr}(\mathbf{d})$  and the gradient of the velocity is obtained by the SPH approximation,

$$\nabla \mathbf{v}_a = \sum_b V_b (\mathbf{v}_b - \mathbf{v}_a) \otimes \nabla w_b(\mathbf{x}_a, h_b) \quad (5)$$

The point integration applied in Equation(2) means that the consistency conditions are no longer satisfied exactly by SPH approximations like in Equation(5). Consistency is enforced by correcting the kernel function and its Hessian by introducing correction factors. Following the theory detailed in [4, 6] the corrected gradient of the kernel function is given by,

$$\tilde{\nabla} \hat{w}_b(\mathbf{x}_a, h_b) := \tilde{\nabla} \hat{w}_b(\mathbf{x}_a, h_b) + \frac{1}{2} \eta \tilde{\mathcal{H}} w_b(\mathbf{x}_a, h_b) \mathbf{h}_a \quad (6)$$

where  $\tilde{\nabla} \hat{w}_b(\mathbf{x}_a, h_b)$  is the linearly corrected gradient of the constant correct kernel function,  $\eta$  is a small constant,  $\mathbf{h}_a = [h_a, h_a]^T$  and  $\tilde{\mathcal{H}} w_b(\mathbf{x}_a, h_b)$  is the linearly corrected Hessian of the kernel function.

## 2. Particle Refinement

In FEM it has been commonplace to use mesh refinement to improve numerical results and reduce computation times. With a corresponding refinement strategy SPH can gain these same advantages. The aim of particle refinement in SPH is to add the ability to combine both sparsely and densely populated regions of particles into the same computational domain.

In order to implement dynamic refinement into a SPH frame work two main considerations need to be dealt with:

- Firstly, identifying a suitable *Criterion for Refinement* as a way to efficiently identify candidate particles for refinement.
- Secondly, to develop a *General Refinement Algorithm* whereby particles are split into a number of corresponding ‘daughter’ particles while preserving the basic properties of the system, **and** minimizing any errors introduced in the process.

Both stages should be applied automatically by an SPH code at runtime with no intervention required. Only particles inside a set region will undergo refinement in this paper. Once a candidate particle is selected for refinement it is necessary to specify the positions, velocities, masses and smoothing lengths for the replacement daughter particles.

In this paper particles are distributed in a hexagonal refinement pattern about the original particle position. The separation of the configuration is governed by the *Separation Parameter*  $\epsilon$  and the new particle smoothing lengths are governed by the *Smoothing Ratio*  $\alpha \in (0, 1]$ . As shown in Figure 1a.

In general the daughter particles also need to be assigned suitable velocities. This will be discussed in Section 2.2. However, in static cases when all the particles are initially stationary, such as a Couette flow, the refined particles velocities can simply be set to zero.

**2.1. Density Refinement Error.** In this section an error measure is defined from the change in local density field due to the refinement procedure. This error is independent of the initial particle’s mass and smoothing length. The optimum daughter particle masses are then calculated for a given distribution through the solution of a constrained minimization problem.

Consider a collection of  $N$  particles. The local density at a point  $\mathbf{x}$  is given by  $\rho(\mathbf{x}) = \sum_{a=1}^N m_a w_a(\mathbf{x}, h_a)$ . Suppose that the  $N^{\text{th}}$  particle is refined into  $M$  daughter particles. The local density distribution will obviously change due to the refinement procedure and is now given by  $\rho^*(\mathbf{x}) = \sum_{a=1}^{N-1} m_a w_a(\mathbf{x}, h_a) + \sum_{b=1}^M m_b^* w_b(\mathbf{x}, h_b)$ . The *Local Refinement Error* at a point  $\mathbf{x}$  is given by,

$$e(\mathbf{x}) := \rho(\mathbf{x}) - \rho^*(\mathbf{x}) = m_N w_N(\mathbf{x}, h_N) - \sum_{b=1}^M m_b^* w_b(\mathbf{x}, h_b) \quad (7)$$

This change in local density field is entirely due to the ability of the refined particles to approximate the contribution of the original unrefined particle.

From the local error the *Global Refinement Error* is given by,

$$0 \leq \mathcal{E} := \int_{\Omega} e(\mathbf{x})^2 d\mathbf{x} \quad (8)$$

Let the masses of the daughter particles be defined by  $m_a^* = \lambda_a m_N$  where  $\lambda_a > 0$  for each  $a$  with conservation of mass enforced by the additional constraint that  $\sum_{a=1}^M \lambda_a = 1$ . Then the global refinement error can be written independently of initial particle smoothing length  $h_N$  and mass  $m_N$  in terms of the unknown coefficients  $\boldsymbol{\lambda} = [\lambda_1, \dots, \lambda_M]^T$  as,

$$\mathcal{E}[\alpha, \varepsilon] = \bar{C} - 2\boldsymbol{\lambda}^T \bar{\mathbf{b}} + \boldsymbol{\lambda}^T \bar{\mathbf{A}} \boldsymbol{\lambda} \quad \text{with} \quad (9)$$

$$\bar{C} = \int_{\Omega} w_N^2(\hat{\mathbf{x}}, 1) d\hat{\mathbf{x}}, \quad \bar{b}_j = \int_{\Omega} w_N(\hat{\mathbf{x}}, 1) w_j(\hat{\mathbf{x}}, \alpha) d\hat{\mathbf{x}}, \quad \bar{A}_{ij} = \frac{1}{\alpha^d} \int_{\Omega} w_i(\hat{\mathbf{x}}, 1) w_j(\hat{\mathbf{x}}, 1) d\hat{\mathbf{x}}$$

The *Model Problem* for a given refinement pattern of  $M$  daughter particles with separation parameter  $\varepsilon$  and smoothing ratio  $\alpha$  is then: To find  $\lambda_j^*$  for  $j = 1, \dots, M$  such that

$$\min_{\boldsymbol{\lambda}} \mathcal{E}[\alpha, \varepsilon](\boldsymbol{\lambda}) = \bar{C} - 2\boldsymbol{\lambda}^{*T} \bar{\mathbf{b}} + \boldsymbol{\lambda}^{*T} \bar{\mathbf{A}} \boldsymbol{\lambda}^* \quad \text{with} \quad \sum_{j=1}^M \lambda_j^* = 1 \quad (10)$$

With the coefficients  $\bar{C}$ ,  $\bar{\mathbf{b}}$  and  $\bar{\mathbf{A}}$  calculated numerically the solution to the above minimization problem is guaranteed since  $\mathcal{E}$  is a convex function. ( $\bar{\mathbf{A}}$  is SPD)

Figures 1b. and 1c. show the resulting optimal approximations to the kernel function when a particle is refined in 1D and 2D respectively.

**2.2. Particle Velocities and Conservation.** It remains to discuss how to assign daughter particle velocities when the particle under refinement is moving with an arbitrary velocity  $\mathbf{v}_N \neq \mathbf{0}$ . Traditionally new particles are given the interpolated velocity from the original unrefined particle configuration. This has the advantage that interpolated velocities follow that of the underlying flow. However they will not in general preserve the kinetic energy, nor the linear and angular momentum of the system.

One such choice that does conserve these properties is to move all daughter particles with the original particle velocity  $\mathbf{v}_N$  that they are replacing. The question remains: Is there a way in which a more representative set of velocities can be chosen such that these global properties are conserved?

In fact the answer is no. To see this consider the particle equations for conservation of kinetic energy and linear momentum respectively,

$$\mathbf{v}_N \cdot \mathbf{v}_N = \sum_{b=1}^M \lambda_b \mathbf{v}_b \cdot \mathbf{v}_b \quad \text{and} \quad \mathbf{v}_N = \sum_{b=1}^M \lambda_b \mathbf{v}_b \quad (11)$$

where the  $\lambda_i$ 's are chosen using the analysis in Section 2.1 with  $\sum_{b=1}^M \lambda_b = 1$ . Suppose that the daughter particles are constrained to move in the same direction as the original particle. In this case  $\mathbf{v}_i = \mu_i \mathbf{v}_N$ ,  $\mu_i > 0$  for each  $i$ . Substituting these velocities into Equation (11) gives two scalar equations in  $\boldsymbol{\mu}$ ,

$$f(\boldsymbol{\mu}) = \sum_{b=1}^M \lambda_b \mu_b^2 - 1 \quad \text{and} \quad g(\boldsymbol{\mu}) = \sum_{b=1}^M \lambda_b \mu_b - 1 \quad (12)$$

The first of the above equations defines an ellipsoid in the  $M$ -dimensional space  $\boldsymbol{\mu}$ , while the second equation defines a plane in the same space. As noted before setting  $\boldsymbol{\mu} = [1, \dots, 1]^T$  satisfies both  $f(\boldsymbol{\mu}) = 0$  &  $g(\boldsymbol{\mu}) = 0$ . It is simple to show that the plane is tangent to the ellipsoid at this point and therefore  $\boldsymbol{\mu} = [1, \dots, 1]^T$  is the **unique** solution that conserves both K.E. and linear momentum.

It is possible to generalize the above result to show this fact is true for the general case in which daughter particles may have arbitrary velocities, but it shall be omitted here.

### 3. Application: Couette Flow

A simple 2D Couette Flow between two infinite horizontal plates is used to test the refinement procedure. The problem domain is a region  $0.0005\text{m} \times 0.001\text{m}$  initially populated by  $20 \times 40 = 800$  evenly distributed particles. The particles in the central region of the flow are then refined in the hexagonal pattern according to the above theory.

The plate separation is  $10^{-3}\text{m}$ , the top plate moves with a constant velocity of  $|\mathbf{v}_T| = 2.5 \times 10^{-5}\text{ms}^{-1}$ . The material density is  $1000\text{kgm}^{-3}$  with kinematic viscosity of  $10^{-6}\text{m}^2\text{s}^{-1}$ .

Figure 2. shows the velocity profile across the flow at several instances in the simulation. The velocities in both the refined and unrefined regions are in good agreement with the analytic solution.

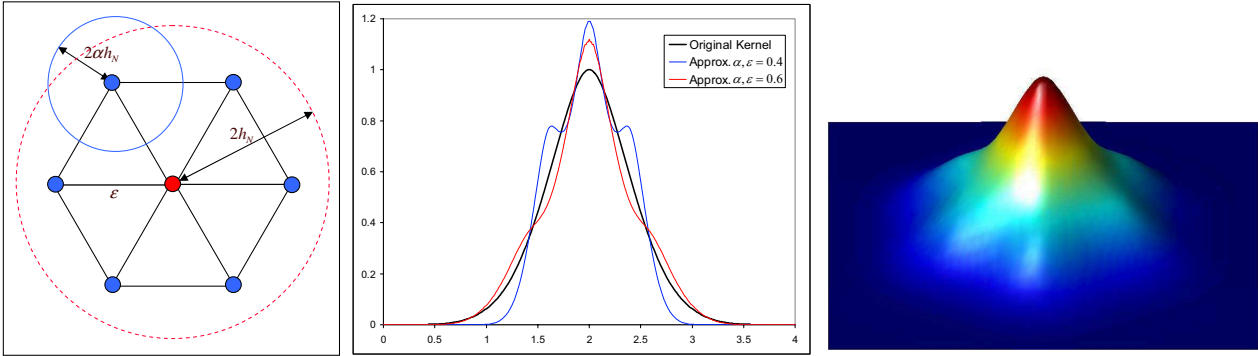


FIGURE 1. a) 2D Hexagonal refinement pattern, b) 1D Kernel approximation with 3 particles, c) 2D Hexagonal kernel approximation

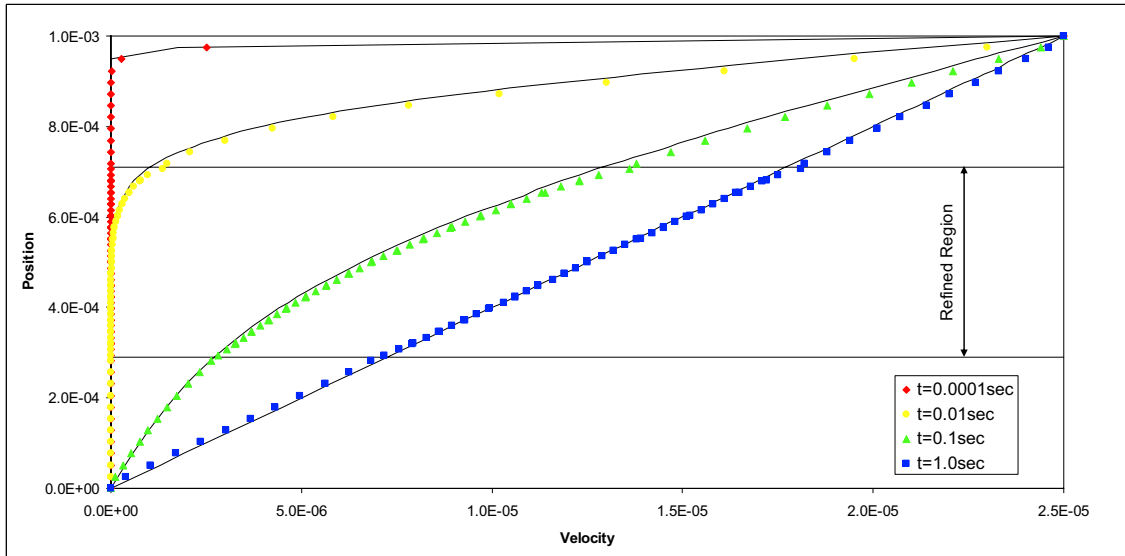


FIGURE 2. Velocity profile of the refined Couette Flow. (series solutions given by the solid lines)

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