# Analytical response sensitivity of laminated composite plate using trigonometric type element

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## 1. Introduction

Fibre reinforced polymeric composite have been used for variety of structural applications, taking advantages of their high specific strength and modulus compared with metals. Initially developed for aerospace industry, high-performance composite now found in application from automotive parts to circuit boards, and from building material to speciality sporting equipment. The primary advantage of composite over their homogeneous isotropic counterparts is the ability to design the physical structure and mechanical properties prior to manufacture. The mechanical behaviour of laminate depend strongly on the direction and thickness of lamina and because of this the lamina should be designed to meet the specific requirement of each particular application in order to obtain the maximum advantages of such material, an accurate and efficient structural analysis, design sensitivity analysis and optimisation procedures are very important to accomplish this task. This paper will present the evaluation of sensitivity of structural response canges in design variables by using trigonometric type element. This finite element model capable to model both thin and moderately thick plates without any pathologies of the classic plate finite elements (shear locking, spurious modes, etc.). It is based on a new kind of kinematics proposed by Touratier [1].

## 2. Laminated Plate Analysis

Consider a $k^{th}$ ply on laminated plate with total thickness $e$, the thickness of each layer $e_i(i = 1, 2, \ldots, n, n$ number of layer), and the boundaries of each layer have the coordinates $h_i(i = 1, 2, \ldots, n + 1)$ as shown in Figure.1, while $\alpha$ is fibers orientation to the $u_1$-axis.

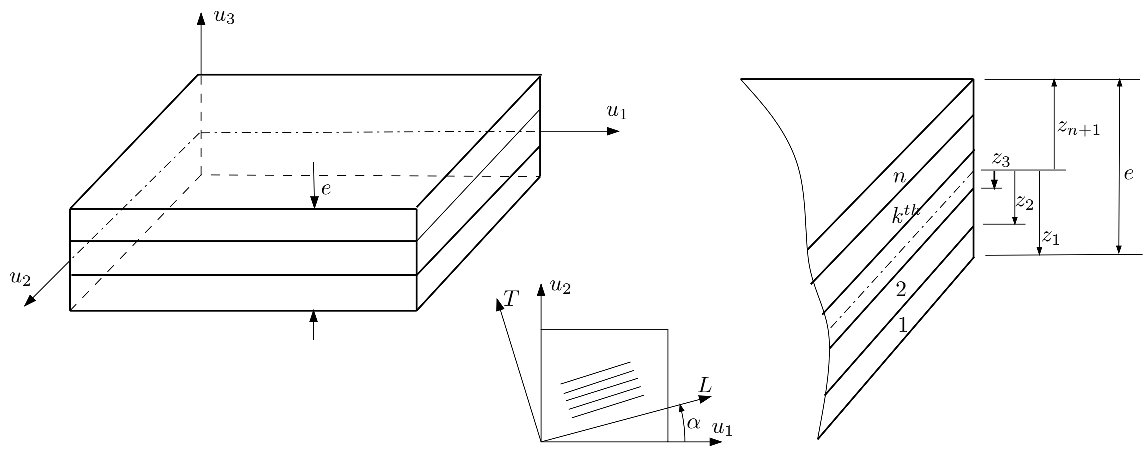


FIGURE 1. n layers laminated composite plate and its geometric property.

We denote by ${u_i}^{(k)}(x, y, z)$, $i \in \{1, 2, 3\}$ the Cartesian components of the displacement field for the $k^{th}$layer of a multilayered plate, and we assume the transverse normal strain denoted by $\epsilon_{33}$ is negligible, according to the moderately thick plate hypothesis. The material behaviour

is admitted linearly elastic and displacements are assumed small in this Section. Strains and stresses are classically denoted by : ${\epsilon_{ij}}^{(k)}$ and ${\sigma_{ij}}^{(k)}$ for the $k^{th}$ layer.
The constitutive law is expressed as :

$$[\sigma_{LT}^{(k)}] = [\widehat{C}^{(k)}][\epsilon_{LT}^{(k)}]$$

$$\text{with } \left\{ \begin{array}{lcl} \widehat{C}_{ij}{}^{(k)} & = & C_{ij}{}^{(k)} - C_{i3}{}^{(k)} C_{j3}{}^{(k)} / C_{33}{}^{(k)} \;\; \text{for } i,j = 1,2,6 \\ \widehat{C}_{ij}{}^{(k)} & = & C_{ij}{}^{(k)} \;\; \text{for } i,j = 4,5 \end{array} \right. \tag{1}$$

where $\widehat{C}_{ij}{}^{(k)}$ are the moduli of the material for the $k^{th}$ layer referred to L and T material's system axis. Relating equation(1) to $u_i$ coordinate system (Fig.1), it becomes

$$[\sigma^{(k)}] = [\bar{C}^{(k)}][\epsilon^{(k)}] \tag{2}$$

where

$$[\bar{C}^{(k)}] = [T^{(k)}]^t[\widehat{C}^{(k)}][T^{(k)}] \tag{3}$$

matrix $[T^{(k)}]$ is tranformation matrix, this matrix is

$$\left[T^{(k)}\right] = \begin{bmatrix} c^2 & s^2 & 0 & 0 & cs \\ s^2 & c^2 & 0 & 0 & -cs \\ 0 & 0 & c & -s & 0 \\ 0 & 0 & s & c & 0 \\ -2cs & 2cs & 0 & 0 & (c^2 - s^2) \end{bmatrix} ; c = \cos\alpha^{(k)}, s = \sin\alpha^{(k)} \tag{4}$$

## 3. Finite Element Formulation

In order to get high quality result from sensitivity analysis, an efficient and accurate six node triangular finite element with shear locking free, trigonometric type element is choosen, proposed by Touratier[1], it is based on the Argyris interpolation [4] for the deflection and the Ganev interpolation [6] for the other generalized displacements. Since the Argyris interpolation is a polynomia of the fifth order and the Ganev interpolation of the fourth order, we can immediately conclude that the transverse shear locking is avoided as the field compatibility is automatically assured for the transverse shear strains, see for example [3]. Note that the Argyris interpolation is precisely of continuity C$^1$, and the Ganev interpolation involves a semi-C$^1$ continuity which is not required here but permits to assure the field compatibility for the finite element approximation of the transverse shear strains.
The elementary stiffness matrix given [1, 2, 3] by:

$$[K_e] = \int_{\Omega_e} [D_e]^T [Tk_e]^T [\Lambda k]^T [A_e][\Lambda k][Tk_e][D_e] d\Omega_e \tag{5}$$

where $[D_e]$ is a geometric transformation matrix between local and global degrees of freedom and , matrix $[\Lambda k]$ is expressed as a function of barycentric co-ordinates, while $[Tk_e]$ only contains geometric constants, so that the product $[\Lambda k]\,[Tk_e]$ may be seen as interpolations with respect to the local degrees of freedom, further detail can be found in [5], and $[A_e]$ is the material behaviour matrix for a multilayered finite element resulting from the integration with respect to the thickness co-ordinate. The exact numerical integration rule for the $[K_e]$ matrix is given in [7] and requires 16 points for the shear part of this stiffness matrix. The same rule has been used for membrane and bending parts.

## 4. Sensitivity Analysis

For static structural analysis, the system equilibrium equation are obtain in the usual way, which is:

$$[K_T][U] = [R] \tag{6}$$

where $[K_T], [U]$, and $[R]$ is system stiffness matrix, system displacement, and system load respectively. Differentiate equation(6) with respect to design variables $b_i$ and rearranged, the result are displacement sensitivity equation wrt design variables $b_i$

$$\frac{\partial[U]}{\partial b_i} = [K_T]^{-1}\left(\frac{\partial[R]}{\partial b_i} - \frac{\partial[K_T]}{\partial b_i}[U]\right) \tag{7}$$

The system stiffness matrix $[K_T]$ is calculated as usual by assembled of element stiffness matrix $[K_e]$ so that

$$[K_T] = \sum_{e=1}^{numel}[K_e] \;; numel \text{ is number of element} \tag{8}$$

differentiate equation(5) with respect to design variables $b_i$ gives

$$\frac{\partial\,[K_e]}{\partial b_i} = \int_{\Omega_e}[D_e]^T[Tk_e]^T[\Lambda k]^T\left(\frac{\partial[A_e]}{\partial b_i}\right)[\Lambda k][Tk_e][D_e]d\Omega_e \tag{9}$$

where

$$\frac{\partial[A_e]}{\partial b_i} = \int_{-e/2}^{e/2}\left(\frac{\partial[B_e]^T}{\partial b_i}[\bar{C}^{(k)}][B_e] + [B_e]^T\frac{\partial[\bar{C}^{(k)}]}{\partial b_i}[B_e] + [B_e]^T[\bar{C}^{(k)}]\frac{\partial[B_e]}{\partial b_i}\right)dz \tag{10}$$

differentiate $[\bar{C}^{(k)}]$ with respect to design variables , will get

$$\frac{\partial\left[\bar{C}^{(k)}\right]}{\partial b_i} = \frac{\partial\left[T^{(k)}\right]^t}{\partial b_i}\left[\widehat{C}^{(k)}\right][T^{(k)}] + [T^{(k)}]^t\frac{\partial\left[\widehat{C}^{(k)}\right]}{\partial b_i}[T^{(k)}] + [T^{(k)}]^t\left[\widehat{C}^{(k)}\right]\frac{\partial\left[T^{(k)}\right]}{\partial b_i} \tag{11}$$

following equation(8) derivative of system stiffness matrix calculated as summation of derivative of element stiffness matrix as

$$\frac{[K_T]}{\partial b_i} = \sum_{n=1}^{numel}\frac{[K_e]}{\partial b_i} \tag{12}$$

As can be seen from equation(10) and (11), computational effort for calculating sensitivity of element stiffness matrix, although sensitivity of matrix $[T^{(k)}], [\widehat{C}^{(k)}]$, and $[B_e]$ need to determined, can be reduced greatly. in the fact, $[T^{(k)}]$ has something to do with the orientation angle the $k$th ply only; $[\widehat{C}^{(k)}]$ is associated with the mechanic property of the material only; $[B_e]$ related to vectorial distance from mid-surface to the boundary of the ply only.

For design variable $b_i$ being orientation of fibers at $k$th ply $\alpha_k$,

$$\frac{[\widehat{C}^{(k)}]}{\partial\alpha_k} = 0 \quad , \frac{[B_e]}{\partial\alpha_k} = 0$$

and for design variable $b_i$ being vectorial distance of the boundary of the ply $z_i$,

$$\frac{[T^{(k)}]}{\partial z_i} = 0 \quad , \frac{[\widehat{C}^{(k)}]}{\partial z_i} = 0$$

Sensitivity with respect to ply thickness $e_i$; $[T^{(k)}]/\partial e_i, [\widehat{C}^{(k)}]/\partial e_i$, and $[B_e]/\partial e_i$ are calculated by means of chain rules

and

$$\frac{[T^{(k)}]}{\partial\alpha_k} = \begin{bmatrix} -\sin(2\alpha_k) & \sin(2\alpha_k) & 0 & 0 & \cos(2\alpha_k) \\ \sin(2\alpha_k) & -\sin(2\alpha_k) & 0 & 0 & -\cos(2\alpha_k) \\ 0 & 0 & -\sin\alpha_k & \cos\alpha_k & 0 \\ 0 & 0 & \cos\alpha_k & -\sin\alpha_k & 0 \\ -2\cos(2\alpha_k) & 2\cos(2\alpha_k) & 0 & 0 & -2\cos(2\alpha_k) \end{bmatrix}$$

bacause of restricted space constrained of this paper and nature of matrix, the $[B_e]/\partial z_i$ not provided here, but can be easily calculated, since matrix $[B_e]$ is provided elsewhere [2, 3].

## 5. Numerical Examples

For illustrative examples of the analytical sensitivity discussed above, a simply supported rectangular laminates plate with dimension 1x3m under uniform load of intensity 1 MPa is analysed. The plate modelled only a quarter to take advantages of symmetry, and uses 32 triangular elements mesh. The ply material properties are $E_L$= 25 GPa, $E_T$= 1 GPa, $G_{12}$=0.5 GPa, $G_{13}$=0.5 GPa, $G_{23}$=0.25 GPA, and $\nu_{12}$=0.25. The examples here use the same material property for each ply. The first example is three ply laminates and arranged the ply so that the orientation is 90/0/90, and second example is five ply laminates and orientated as 90/0/90/0/90. The result of analytical compared with forward finite different method (FFD) with perturbation values 1% for thickness and $0.1^o$ for angle, the result are shown in the tables below.

Table 1. Deflection sensitivity for three layer [90/0/90] laminated plate

sensitivity wrt ply angle $\partial U/\partial\alpha$

| Layer | Analytic | FFD |
|---|---|---|
| 1 | -0.11582 | -0.11596 |
| 2 | 0.08851 | 0.08856 |
| 3 | -0.11582 | -0.11596 |

sensitivity wrt ply thickness $\partial U/\partial e$

| Layer | Analytic | FFD |
|---|---|---|
| 1 | -5.13685 | -5.13812 |
| 2 | -13.59093 | -13.77844 |
| 3 | -5.13685 | -5.13812 |

Table 2. Deflection sensitivity for five layer [90/0/90/0/90] laminated plate

sensitivity wrt ply angle $\partial U/\partial\alpha$

| Layer | Analytic | FFD |
|---|---|---|
| 1 | -0.01203 | -0.01206 |
| 2 | 0.04321 | 0.04325 |
| 3 | 0.00134 | 0.00133 |
| 4 | 0.04321 | 0.04325 |
| 5 | -0.01203 | -0.01206 |

sensitivity wrt ply thickness $\partial U/\partial e$

| Layer | Analytic | FFD |
|---|---|---|
| 1 | -0.46793 | -0.46240 |
| 2 | -2.88214 | -2.91320 |
| 3 | -2.54348 | -2.57060 |
| 4 | -2.88214 | -2.88720 |
| 5 | -0.46793 | -0.46240 |

## 6. Conclusion

The result obtained show that sensitivity analysis derived from trigonometric (sinus) model finite with respect to angle orientations and ply thicknesses proved to be and efficient, by comparation with the sensitivities calculated by global finite different method, and reveal promising applicability in the optimisation process.

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