Limit analysis using large-scale SOCP optimization

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1. Introduction

Limit analysis is a useful tool for estimating the load-carrying capacity of a structure. The use of the static and kinematic theorems in combination with the finite element method allows us to bracket the exact value of the limit load. In general, however, a close bracketing can be achieved only if we use a huge number of finite elements, and thus an effective algorithm for large and sparse optimization is absolutely necessary. Here we discuss the formulation of limit analysis as a second-order cone programming (SOCP) problem. This is very attractive since SOCP has enjoyed great attention from mathematicians (the quadratic cones are self-dual, and thus efficient primal-dual interior-point algorithms can be derived). Compared with general nonlinear programming, a major advantage of using SOCP for limit analysis is that we can overcome the usual difficulties like the singular Hessian matrix of the yield function, and (for frictional materials) the non-differentiability of the yield surface at its apex.
Concerning the lower bound (LB) method, a brief description has been given in reference [6]. The emphasis here is on the upper bound (UB) method. For a rigorous UB analysis, the main difficulty is choosing a finite number of flow rule points so as to enforce the flow rule over the whole area of each element. This problem can be overcome for incompressible materials, however for cohesive-frictional materials the only available choice until now has been the use of constant strain elements combined with kinematically admissible discontinuities [4]. For cone-shaped yield criteria (where the UB problem can readily be formulated in terms of kinematic quantities only) we point out that the six-node triangle with straight sides is also a suitable choice. Due to its higher order, this element is more suitable for unstructured meshes than the one of reference [4], where good performance appears to be quite dependent on the topology of the discontinuities. Another important advantage of the six-node element is that it can avoid the locking problem that was detected by Nagtegaal *et al.* [10]. To close the paper we analyse a difficult numerical example using a state-of-the-art SOCP algorithm; this shows that in practice our formulation is extremely advantageous from the aspects of speed, stability and accuracy. To the authors' knowledge these are the largest optimization problems (some with over 340000 variables) that have ever been solved in the area of limit analysis.
Although the application of SOCP is restricted to yield functions that can be expressed in a conic form, this includes a variety of the most popular yield criteria such as Mohr-Coulomb in plane strain, Drucker-Prager in 3D (consequently von Mises), Nielsen's criterion for plates and Ilyushin's for shells. Some previous work employing SOCP for shakedown analysis can be found in references [2, 5] where the von Mises yield restriction was used.

2. Second-Order Cone Programming

A SOCP problem (also referred to as conic quadratic optimization, CQO) has the form

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x}^i \in \mathcal{C}_i \quad \forall i \in \{1,\dots,N\} \\ & \mathbf{x}^T = \left[(\mathbf{x}^1)^T \dots (\mathbf{x}^N)^T\right] \end{aligned} \tag{1}$$

with $\mathbf{A} \in \Re^{m\times n}$, $\mathbf{b} \in \Re^m$, $\mathbf{c}, \mathbf{x} \in \Re^n$, $\mathcal{C}_i = \{\mathbf{x}^i \in \Re^{d_i} : \|\mathbf{x}^i_{2:d_i}\| \le x^i_1, \; x^i_1 \ge 0\}$. The sets $\mathcal{C}_i$ are second-order (or quadratic) cones. In (1) it has been assumed that there are no free variables. For brevity in what follows we will also employ the notation $(z, \mathbf{x}) \in \mathcal{C}$ meaning $\|\mathbf{x}\| \le z$.

3. Yield criterion and dissipation function

It is convenient to use stresses and strains in the deviatoric form, *i.e.*

$$\sigma_{\rm m} = \frac{1}{N}\sum_{i=1}^{N}\sigma_{ii}, \quad s_{ij} = \sigma_{ij} - \sigma_{\rm m}\delta_{ij}, \quad \theta = \sum_{i=1}^{N}\varepsilon_{ii} \quad \text{and} \quad \epsilon_{ij} = \varepsilon_{ij} - \frac{1}{N}\theta\delta_{ij} \tag{2}$$

where N is the dimension of the tensors ($N = 2$ in plane strain) and δ is Kronecker's delta. The Mohr-Coulomb yield criterion in plane strain is

$$\sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2} + (\sigma_{11} + \sigma_{22})\sin\phi \le 2c\cos\phi \tag{3}$$

where c is the cohesion and ϕ the friction angle. After taking advantage of $s_{22} = -s_{11}$ this yield restriction can take the form

$$\sqrt{s_{11}^2 + s_{12}^2} + a\sigma_{\rm m} - k \le 0 \quad \text{or} \quad f(\sigma_{\rm m}, \mathbf{s}^{\rm red}) = \|\mathbf{s}^{\rm red}\| + a\sigma_{\rm m} - k \le 0 \tag{4}$$

where $a = \sin\phi$, $k = c\cos\phi$ and $\mathbf{s}^{\rm red} = \begin{bmatrix} s_{11} & s_{12}\end{bmatrix}^T$. This means that the set of plastically admissible stresses $K_Y = \{\boldsymbol{\sigma} : f(\boldsymbol{\sigma}) \le 0\}$ is a second-order cone set

$$(k - a\sigma_{\rm m}, \mathbf{s}^{\rm red}) \in \mathcal{C}$$

Defining the dissipation function

$$d_p(\boldsymbol{\varepsilon}) = \max_{\boldsymbol{\sigma}\in K_Y}\sum_{i,j}\sigma_{ij}\varepsilon_{ij} \tag{5}$$

it can be proved that

$$d_p(\boldsymbol{\varepsilon}) = d_p(\lambda, \boldsymbol{\epsilon}^{\rm red}) = k\lambda \quad \textit{iff} \quad (\lambda, \boldsymbol{\epsilon}^{\rm red}) \in \mathcal{C}, \;\; \theta = a\lambda \tag{6}$$

where $\boldsymbol{\epsilon}^{\rm red} = \begin{bmatrix} 2\epsilon_{11} & 2\epsilon_{12}\end{bmatrix}^T$. All of the above can be generalized for the Drucker-Prager yield criterion in 3D, which has the form

$$\sqrt{J_2(\mathbf{s})} + a\sigma_{\rm m} \le k \tag{7}$$

where a and k are material constants. In this case for the dissipation we have

$$d_p(\boldsymbol{\varepsilon}) = k\lambda \quad \textit{iff} \quad 2\sqrt{J_2(\boldsymbol{\epsilon})} \le \lambda, \;\; \theta = a\lambda \tag{8}$$

4. Application of the kinematic theorem

Let us consider a rigid–perfectly plastic structure $V \in \Re^3$ with boundary $\partial V = S_u \cup S_t$ and $S_u \cap S_t = \emptyset$. Displacements $\mathbf{u}_0$ are prescribed on S_u, and surface tractions $\mathbf{t}(\mathbf{x})$ on S_t. According to the kinematic theorem, among all displacement fields $\mathbf{u}$ that satisfy the boundary conditions on S_u and the flow rule throughout V, the limit load multiplier β of the surface tractions can be calculated from

$$\begin{aligned} \min \quad & D_p(\boldsymbol{\varepsilon}) - \int_V \boldsymbol{g}^T\mathbf{u}dV \\ \text{s.t.} \quad & \int_{S_t}\mathbf{t}^T\mathbf{u}dS_t = 1 \end{aligned} \tag{9}$$

where $D_p(\boldsymbol{\varepsilon}) = \int_V d_p(\boldsymbol{\varepsilon})dV$, $\varepsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ and $\boldsymbol{g}$ = body force vector per unit volume.

According to the previous section and considering deviatoric strains, the problem becomes

$$
\begin{aligned}
\min \quad & \int_V k\lambda dV - \int_V \boldsymbol{g}^T \mathbf{u} dV \\
\text{s.t.} \quad & \int_{S_t} \mathbf{t}^T \mathbf{u} dS_t = 1 \\
& \mathbf{u} \in U \\
& \epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{N}\delta_{ij}\text{div}\mathbf{u} \quad && \forall \mathbf{x} \in V \\
& \text{div}\mathbf{u} = a\lambda && \forall \mathbf{x} \in V \\
& 2\sqrt{J_2(\boldsymbol{\epsilon})} \leq \lambda && \forall \mathbf{x} \in V
\end{aligned}
\tag{10}
$$

where $U = \{\mathbf{u} : \mathbf{u} = \mathbf{u}_0, \quad \forall \mathbf{x} \in S_u\}$. For an FEM discretization in plane strain, the final optimization problem takes the form

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{NP} \bar{\mathbf{k}}_i \mathbf{z}_i - \mathbf{q}_g^T \mathbf{u} \\
\text{s.t.} \quad & \mathbf{q}_s^T \mathbf{u} = 1 \\
& \mathbf{u} \in U \\
& \mathbf{A}\mathbf{z}_i = \mathbf{B}_i \mathbf{u} \quad && \forall i \in \{1, \ldots, NP\} \\
& \mathbf{z}_i \in \mathcal{C}_i && \forall i \in \{1, \ldots, NP\}
\end{aligned}
\tag{11}
$$

where

$$
\bar{\mathbf{k}}_i = \begin{bmatrix} kA_i^e/3 & 0 & 0 \end{bmatrix}, \quad \mathbf{z}_i^T = \begin{bmatrix} \lambda_i & (\boldsymbol{\epsilon}_i^{\text{red}})^T \end{bmatrix}, \quad \mathbf{A} = \text{diag}\begin{bmatrix} a & 1 & 1 \end{bmatrix},
$$

$\mathbf{q}_s$ and $\mathbf{q}_g$ are the surface traction and self-weight nodal load vectors, $\mathbf{B}_i$ is related to the usual strain–displacement matrix, and A_i^e is the area of the element to which the flow rule point i belongs. Also NP is the total number of flow rule points, and should be such that the flow rule is satisfied throughout each element. This can occur if we choose 6-node triangular elements with straight sides, with the three vertices being the flow rule points (the strains can then be expressed as a simplex; in 3D 10-node tetrahedral elements with plane faces could be used). Finally, instead of solving the problem (11) it is more convenient to solve the dual.

5. Numerical example

We consider the bearing capacity of a vertically loaded, rigid strip footing on purely frictional soil without surcharge. As noted in [4] and many previous studies, a very fine mesh (especially near the edge of the footing) is needed to obtain accurate results for this benchmark. In the LB analyses the mesh was semi-structured, with a fan of 80 triangles at the footing edge. This mesh contained 24163 elements, and included extension elements in order to model the semi-infinite soil mass. In the UB analyses the meshes were absolutely unstructured, but again very fine close to the footing edge. They consisted of 30778 and 37802 elements for smooth and rough footings respectively. The meshes were prepared using GID [3]. The analyses were performed on a Dell PC (2.66 GHz CPU, 2 GB RAM) in the WinXP environment, using the conic optimizer of the MOSEK software package. This algorithm is based on the interior-point method (see reference [1]) and has been shown to be highly efficient.
Table 1 gives the results, expressed in terms of the usual dimensionless factor $N_\gamma = 2q_\text{u}/\gamma B$, and compares them with the exact plasticity solutions presented in references [7, 8]. The CPU times in the table do not include the time spent reordering of variables and rows (about 20 s for LB; 20 and 30 s for smooth UB and rough UB respectively). It is noteworthy that all values in Table 1 are consistent with the relevant exact solution (LB smaller, UB greater); this is not the case for the bounds obtained in [4]. Finally, the short CPU times for these large problems (over 217000 variables in the LB analyses and 340000 in the rough UB analyses, *i.e.* the largest discrete limit analysis problems that have ever been solved) show the great benefits that can be gained by the use of SOCP and modern interior-point algorithms in general.

	LOWER BOUNDS				UPPER BOUNDS			
	smooth		rough		smooth		rough	
ϕ (°)	result (error%)	CPU (s) (iter)	result (error%)	CPU (s) (iter)	result (error%)	CPU (s) (iter)	result (error%)	CPU (s) (iter)
5	0.08415 (-0.37)	56 (24)	0.1119 (-1.30)	57 (22)	0.08538 (1.08)	87 (22)	0.1221 (7.70)	111 (22)
10	0.2800 (-0.31)	68 (29)	0.4273 (-1.36)	67 (27)	0.2841 (1.15)	87 (22)	0.4513 (4.19)	135 (27)
15	0.6973 (-0.37)	75 (32)	1.166 (-1.26)	67 (29)	0.7073 (1.06)	103 (26)	1.213 (2.68)	135 (27)
20	1.575 (-0.26)	76 (32)	2.811 (-0.98)	75 (32)	1.598 (1.23)	113 (29)	2.898 (2.07)	141 (28)
25	3.452 (-0.27)	92 (37)	6.447 (-0.69)	77 (33)	3.507 (1.31)	113 (29)	6.630 (2.14)	165 (33)
30	7.630 (-0.30)	83 (36)	14.67 (-0.55)	84 (36)	7.766 (1.47)	105 (27)	15.06 (2.06)	157 (31)
35	17.51 (-0.37)	92 (39)	34.23 (-0.71)	98 (42)	17.87 (1.68)	110 (28)	35.23 (2.19)	188 (37)
40	42.97 (-0.51)	92 (40)	84.73 (-0.98)	104 (45)	43.99 (1.87)	121 (31)	88.15 (3.02)	200 (40)
45	116.8 (-0.68)	96 (42)	231.4 (-1.22)	116 (50)	120.3 (2.31)	117 (30)	245.7 (4.89)	230 (46)

Table 1. Bounds on N_γ for various friction angles, for smooth and rough footings

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