R-adaptive and h-adaptive mesh optimisation based on error assessment

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1. Introduction

The finite element method is one of the most powerful techniques for finding approximate solutions to the differential equations that we encounter in engineering problems. In this approach, a continuous system is divided into a discrete physical representation consisting of a finite number of elements. This subdivision of the continuum will result in a discretisation error. This discretisation error is associated with discontinuities or jumps in the stresses or strains between elements. An estimation of the true/exact discretisation error is important to evaluate and improve finite element results. One of the approaches to estimate these errors is smoothing techniques. This approach evaluates how well the approximate solution satisfies continuity of stresses and strains. Two pointwise smoothing error measures will be used, namely the absolute and the relative measures of discretisation error [1]. The value of the first measure depends solely on the level of the discretisation error contained in each element regardless of the level of stress. The other measure is a relative measure of discretisation error. Next, the estimated error can be used to improve or optimise the finite element mesh. This process is known as mesh adaptivity. R adaptivity and H adaptivity are the focus of this paper. R-adaptivity is based on relocating the nodes by maintaining the same number of degrees of freedom and element connectivity. As a consequence, the computational cost remains low. In H- adaptive strategy, the connectivity of the elements as well as the total number of degree of freedom may change while the polynomial degree of the shape functions remains the same. Further, it is computationally expensive but easy to implement [3, 5].
In the case of r- adaptivity, a new formula will be derived from Li and Bettess (LB) remeshing strategy which enables us to calculate the element size from the estimated error. In the case of h- adaptivity, the Li and Bettess remeshing strategy will be used to calculate the desired element sizes from the estimated error by taking into account a certain prescribed accuracy. Furthermore, the evaluation of a selected error estimator will be studied through these remeshing strategies by another error estimator. We will compare the effectiveness of r- and h- adaptive strategies in eliminating the errors in re-entrant corners with a stress singularity. Finally, a conclusion will be drawn regarding the reliability and capability of the r- and h- adaptive strategies as well as the reliability of different error estimators.

2. R- and h-adaptive procedures

By means of a remeshing strategy, the element-wise computed estimated error can be translated into improved discretisation. A pointwise error estimator has been used to measure the discretisation errors as a function of the differences between the finite element result and the smoothed stresses at the nodes [1]. Two different procedures to measure discretisation errors are studied, namely a relative measure where the errors are scaled with the absolute values of the maximum finite element stresses existing in the overall problem and an absolute measure where the scaling will be made with respect to the maximum stresses existing within each element. Next, for each element it must be determined what the desired element sizes

should be. To this end, an optimality criterion is needed, as well as an error tolerance in h-adaptivity. Finally, an algorithm must be derived by which these desired element sizes can be realised. In h-adaptivity, a mesh generator can be used. For r-adaptive schemes, the use of an equidistribution equation leads to optimal meshes. Here, the optimality criterion according to Li and Bettess (LB) is used [6].
The new formula has been driven from the Li and Bettess (LB) remeshing strategy by taking into account that the number of elements remains invariant in the case of r-adaptive strategy. The derived formula for a two dimensional case using six nodes triangular elements will be

$$RR = \left[\frac{||e||_k}{||\hat{e}||} \left(\sum_{k=1}^{M} \left(\frac{||e||_k}{||\hat{e}||} \right)^{2/3} \right)^{3/4} \right]^{1/3} \tag{1}$$

where RR is the refinement ratio defined as the ratio of current element size and desired element size. The next step as mentioned is to attain an equidistribution of the relocating indicator RR and this can be achieved with solving the following system of non-linear equations [2, 5]:

$$\frac{\partial}{\partial X^{cur}} \left(RR \cdot \frac{\partial X}{\partial X^{cur}} \right) = 0 \tag{2}$$

where X^{cur} is the current nodal position, and X are the nodal coordinates which are the fundamental unknown of the equation. The global error that will be used as termination criteria for both the relative and absolute pointwise measures can be obtained from

$$||\hat{e}|| = \frac{\left[\sum_{k=1}^{M} \left(||e||_k \right)^{2/3} \right]^{3/2}}{\hat{M}} \tag{3}$$

In the case of h-adaptivity, the Li and Bettess remeshing strategy will be used where we calculate the number of elements in the new mesh for a desired accuracy that we demand. Next, we calculate the number of elements of the new mesh laying inside the element of the original mesh, this value can be obtained from

$$\left[\frac{h_k}{\hat{h}_k} \right]^2 = \left[\frac{\sqrt{\hat{M}} \cdot ||e||_k}{||\hat{e}||} \right]^{2/3} \tag{4}$$

3. Examples

We will show two examples; the first one demonstrates the r-adaptive algorithm and the second one illustrates the use of the h-adaptive algorithm. Both are based on relative ratio measures. Next, a comparison will be made with results achieved for both algorithms using the absolute measure.

3.1. **R-adaptive algorithm.** The first example deals with a cantilever beam subjected to a concentrated load ($P = 100$ kN) at the free end. The beam has a dimension of (4×8) and contains two internal holes (1×1). We apply the r-adaptive remeshing strategy according to the relative measure on a uniform mesh (Figure 1.A, nr elem = 396, nr nodes = 864). As we take a closer look on the results, it clear that the relative measure identifies the element under the load as well as those at the corners of the internal holes where the highest errors occurs as being in the most need of refinement (Figure 1.B). The global error decreases from its initial value 17.5 to 4.87 at the fourth iteration. On the other hand we have an increase in the global error for the corresponding values of the absolute measure. The question is why the relative pointwise measure converges with a very low number of elements, while the absolute measure fails to do that?
As we know, the relative pointwise measure depends on the stress level in the individual elements. As the mesh is refined in the critical parts, the strain gradient will increase. This

A. Original mesh B. Second iteration

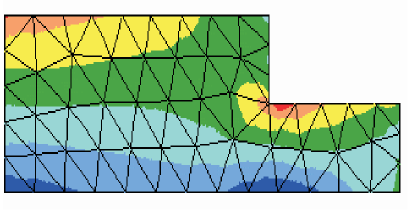
FIGURE 1. R-Adaptive Strategy based on the relative measure

would increase the discretisation error in the elements responsible for representing the increasing strain gradient that will result in a better representation of the stresses. This can easily be recognised within the element under the discontinuous load where the stress levels are increasing throughout the iteration process. The values of these stresses are much higher than the stresses around the internal holes or those at the rigid end and significantly higher than those in the rest of the structure. The denominator of the relative measure is given by summing the absolute values of the maximum finite element stresses existing in the overall structure. As a result of the previous discussion, the significant increase in the stresses (the denominator of the relative measure) in the critical parts of the structure throughout will dominate the results and makes it look like as it is converging throughout the iteration process.
In the case of r-adaptive strategy based on the absolute refinement ratio measure, the global error will increase throughout the iteration process, since the strain gradient will increase as the mesh is refined along the critical parts of the structure. This would increase the discretisation error in the elements responsible for representing the increasing strain gradient that will result in a better representation of the stresses. The rest of the structure will be assigned a coarser mesh due to the refinement along the critical parts. This means a high level of errors in the whole structure. As a result the global error will increase and this makes the r-adaptive strategy not suitable to be used in eliminating pure discretisation errors.

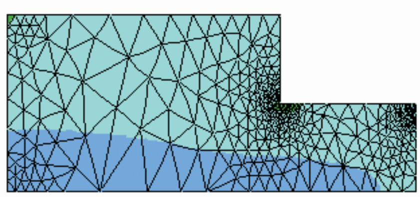
3.2. **H-adaptive algorithm.** Secondly, we study a cantilever beam (dimensions: 6×2) with an external hole (dimensions: 2×1). The beam is subjected to a concentrated load at the free end ($P = 100$ kN). The beam has a uniform mesh (Figure 2.A, nr. node=211, nr elem=86). We apply the h-adaptive strategy (prescribed error=10%) based on the relative measure (Figure2.B, 2C). The global error meets the required error within one iteration and drops off to 0.465% at the sixth iteration while the prescribed error remains unchanged (Figure 2D). Although we have almost total convergence within the sixth iteration, the absolute measure does not recognise this convergence. The corresponding results of the absolute measure show either an increase or decrease in the level of errors in the structure depending on how uniform and intensive the elements are around that region. The same explanation given in the previous example can be given here. The only difference is that the increase in the level of stresses in the critical parts will be much higher due to the characteristic of h-adaptivity where more elements can be added. This will give a much higher value in the denominator of the relative measure resulting in a higher rate of convergence.
Finally, using h-adaptive algorithm based on absolute refinement ratio measure leads to eliminating pure discretisation errors everywhere in the structure and containing the errors in the singularity elements. This procedure approaches continuity through the decreasing range of stresses in which the whole structure is enclosed by. The procedure is capable to meet the required prescribed error within one iteration. Nevertheless, the procedure is computationally expensive due to the required number of elements to be added.

4. CLOSURE

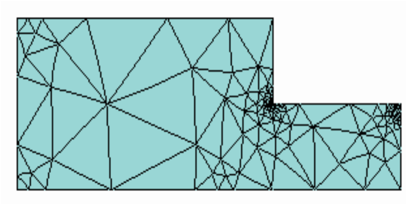
As has been presented in the previous section we can conclude that an r-adaptive strategy based on relative measures is capable in improving the stress representation in the most



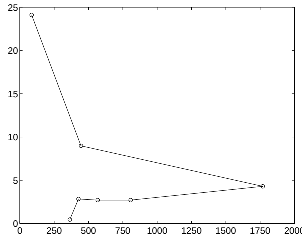
A. Original mesh, nr. elem. = 86



B. Second iteration, nr. elem. = 1769



C. Fourth iteration, nr. elem. = 567



D. Relative global error versus number of elements

FIGURE 2. H-Adaptive Strategy based on the relative measure

critical parts of the structure (parts with high level of stresses). This will result in higher pure discretization errors in these parts. The remaining parts of the structure will be discretised with a coarser mesh due to the refinement in the critical parts. This means a higher level of errors in the in the whole structure. We can consider the r-adaptive strategy as unreliable method in eliminating pure discretisation errors. Further we have seen that the relative pointwise measure is unreliable and produces deceiving results of the estimated discretisation errors. The element with the highest level of relative error (highest level of stresses) will undermine the level of discretisation errors for all the elements in the structure according to their stress levels.

We have also seen misleading results of convergence using h-adaptive strategy based on the relative pointwise measure. This procedure will lead either to a relative increase or decrease in pure discretisation error. In the case of h-adaptive strategy based on the absolute measure, the procedure is capable of containing the discretisation errors in the singularity elements and eliminating them effectively for the non singularity elements. In both procedures, the prescribed error has been met within the first iteration.

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