Limit analysis applied to cantilever stairs

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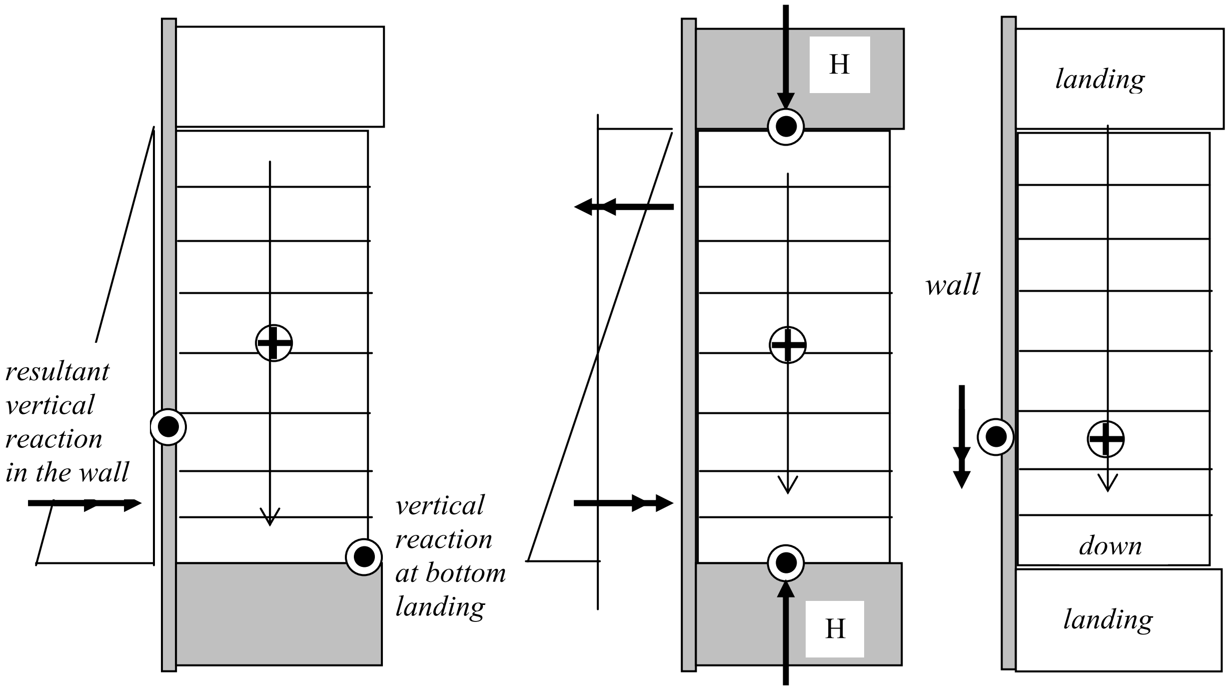
1. INTRODUCTION

This paper addresses the problem of assessing the stability and load capacity of a masonry structure subject to limits placed on contact stresses between components. The particular form of structure considered is that of a so-called cantilever staircase where the treads are formed from individual pieces of stone which appear to cantilever from a supporting wall, but also are in contact with each other along waist sections [1,2]. Questions of stability can arise as recently reported [3]. The model proposed in [2] for this type of structure involves essentially well defined statically determined force paths involving three basic modes of action which are illustrated in simplified diagrammatic plans in Figure 1. The shaded areas indicate active support structures for the flight of stairs.



(1) (2) (3)

FIGURE 1. Modes of action for a straight flight of stairs.

1. simply supported treads, i.e. simply supported at the wall end and simply supported on the tread below at its other end. Forces are transmitted down the flights with torsional reactions from the wall to provide rotational stability. Distributions of torsion are indicated in Figures 1(1) and 1(2).
2. supplemented by some degree of horizontal interaction between the treads in the manor of an arch, e.g. constant horizontal force H which can halve the maximum torsional reaction.
3. cantilever action by independent treads, though this may be considered as a "last resort" since the mobilisation of bending moment reactions at the wall depends on the manner of connection and the capacity of the wall. Load and reaction vectors for a typical tread are shown in Figure 1(3).

In this paper only modes (2) and (3) are combined treating the arching action in mode (2) in a similar way to a flying buttress, with the treads considered as voussoirs and the landings as abutments, but with the addition of torsional reactions to each tread. These enable the thrust line to become discontinuous so as to pass through the relatively thin waist sections as illustrated in Figure 2. The arch mode is then statically indeterminate of degree 3 for the arch ring plus 1 torsional reactive moment per tread – the latter being split into 2 components involving vertical and horizontal interfaces with the wall. Limit analysis is implemented for a straight flight assuming the reactions from the wall and landings and the interactions between treads are only constrained by contact stresses and friction. It is further assumed for the current paper that the stresses within the treads due to bending and torsion are insignificant. The constraints are posed in terms of stress-resultants at supports and between treads which form piecewise quadratic convex limit surfaces – as illustrated in Figure 3 for tread interactions in non-dimensional form. The variables are taken as a set of stress-resultant biactions which form the basis for the space of hyperstatic solutions with dimension equal to the degree of statical indeterminacy of the skeletal model.

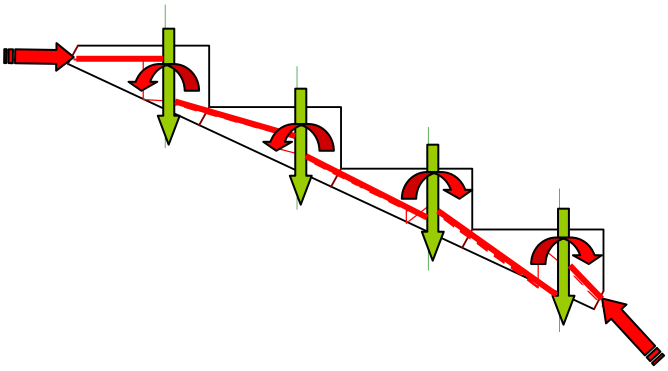


FIGURE 2. A discontinuous thrust line for arching action

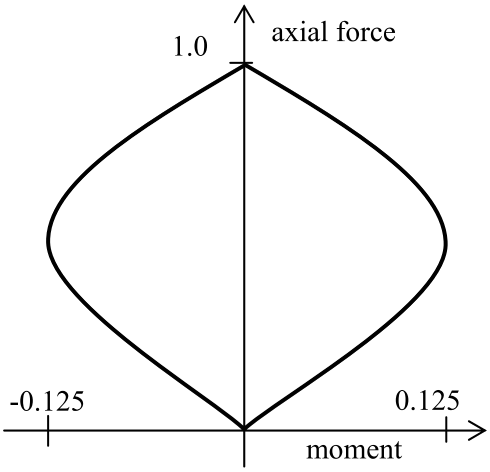


FIGURE 3. Tread interaction diagram

2. LIMIT ANALYSES

The form of limit analysis considered is one of maximising the live load factor λ as indicated in Equation (1), subject to non-linear constraints on contact stresses.

$$W_{Total} = W_{Dead} + \lambda \cdot W_{Live} \qquad (1)$$

Due to the non-linear nature of some of the constraints there is a choice for solution method between using linearised constraints in a linear programme, e.g. [4], or to use a non-linear programme. The Excel spreadsheet has been used since it has the facility to invoke a non-linear generalized reduced gradient (GRG) algorithm which appears to be well suited to the problem – the non-linear constraints being well defined in convex quadratic forms.

Numerical examples are based on straight flights of granite stairs within Castle Drogo [5] which consist of just 4 treads between landings as indicated in Figure 2. Results are presented to illustrate the convergence of the iterative procedure from different initial solutions, and the potential problems of distinguishing between local and global optima. Due to the inherent uncertainties in quantifying some of the parameters, the results of sensitivity analyses are also given to illustrate the benefits of this type of limit analysis to the assessment of historic structures.

3. CONVERGENCE STUDIES

Figures 4 and 5 illustrate typical convergence behaviour when the bearing stresses in the wall are limited to 0.5 N/mm^2, the contact stresses between treads are limited to 1.0 N/mm^2, and the friction coefficient $\mu = 0.5$. The live load is restricted to the second tread down from the top landing. The dead

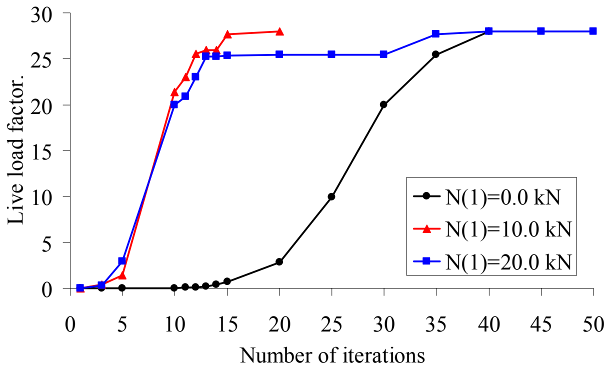


FIGURE 4. Convergence of λ

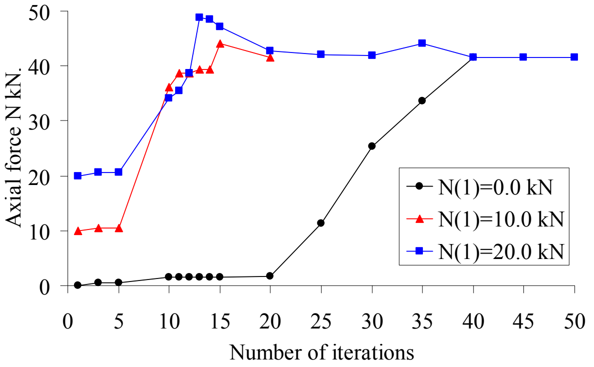


FIGURE 5. Convergence of axial force N_3

load for each tread is taken as 1.042kN, and the unfactored live load is taken as 1.0kN. The total number of static variables is 3 + 3×4 = 15, with 22 constraints. All variables are initially set to zero except for the axial force N_3 in the mid-span waist section. This force serves as a measure of the arching action. The initial value has the physical significance of an axial prestressing force which is then allowed to vary. The initial values are shown as $N(1)$ in Figures 4 and 5.

The operational parameters of GRG were set to an absolute value for the constraint tolerance of 10^{-6}, and a relative value of 1 in 10^{6} for the convergence criterion, i.e. the iterations stop when the relative changes in the objective function λ are less than 10^{-6} over 5 iterations. These parameters lead to the same solutions within 45 iterations with monotonic convergence, although as can be seen in Figure 4 the solution initiated with $N(1) = 20$ kN converged to a lower value when the convergence criterion was raised to 10^{-5}. Clearly the value of initial prestress has a significant effect on the number of iterations required. However as soon as $\lambda > 0$, the self weight becomes supportable and all subsequent solutions are feasible even if not optimal. This reflects one of the benefits of an optimization method based on equilibrium.

Figure 5 shows non-monotonic convergence of the axial force from various initial values. When $N(1) = 0$, almost half the number of iterations is needed to mobilize significant arching action. A check on convergence is provided by the evidence in the spreadsheet that arching and cantilever actions are fully mobilized and only the frictional constraints may not be attained. The final solution shows $\lambda = 28.0$ and only 23.3% of the total load is supported by the treads as cantilevers, the remaining 76.7% being supported by the arch.

4. SENSITIVITY STUDIES

A number of parameters, e.g. the depth d of embedment of the treads, the limiting bearing and contact stresses, and the friction coefficient between the treads are likely to have uncertain values. In which case it is prudent to consider ranges of values in sensitivity analyses, and the results of typical analyses

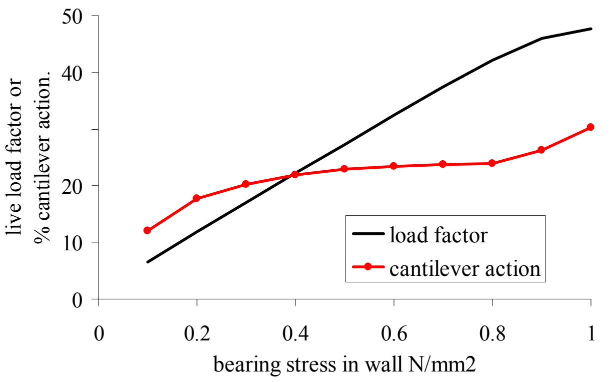


FIGURE 6. sensitivity wrt wall bearing stress

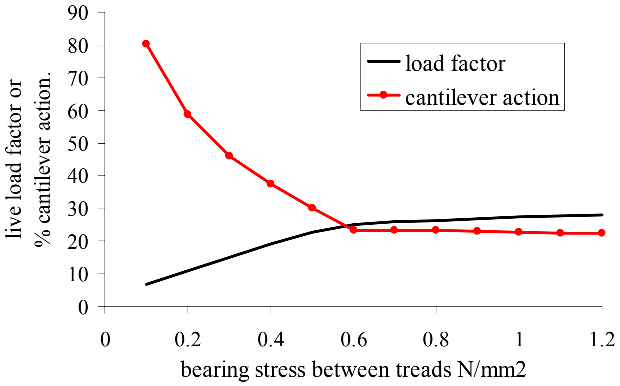


FIGURE 7. Sensitivity wrt tread contact stress

for the staircase in Castle Drogo are illustrated in Figures 6 and 7.

5. CURVED FLIGHTS OF STAIRS

Some of the most beautiful stone stairs are those which are spiral or oval in plan form. Mode (1) has also been proposed [2] for these, but long flights including short landings are common and this mode raises questions re the extreme values of torsion and vertical reaction at the bottom tread. It is thus proposed to extend the type of limit analysis used above to include all three modes of action for long curved flights including intermediate landings. This requires the concept of a 3D curved arch with the enclosing wall providing both torsional reactions AND a radial thrust (wrench) to bend the main thrust line along a spiral trajectory. The extended system of forces and torsion is illustrated in Figure 8 for a typical tread.

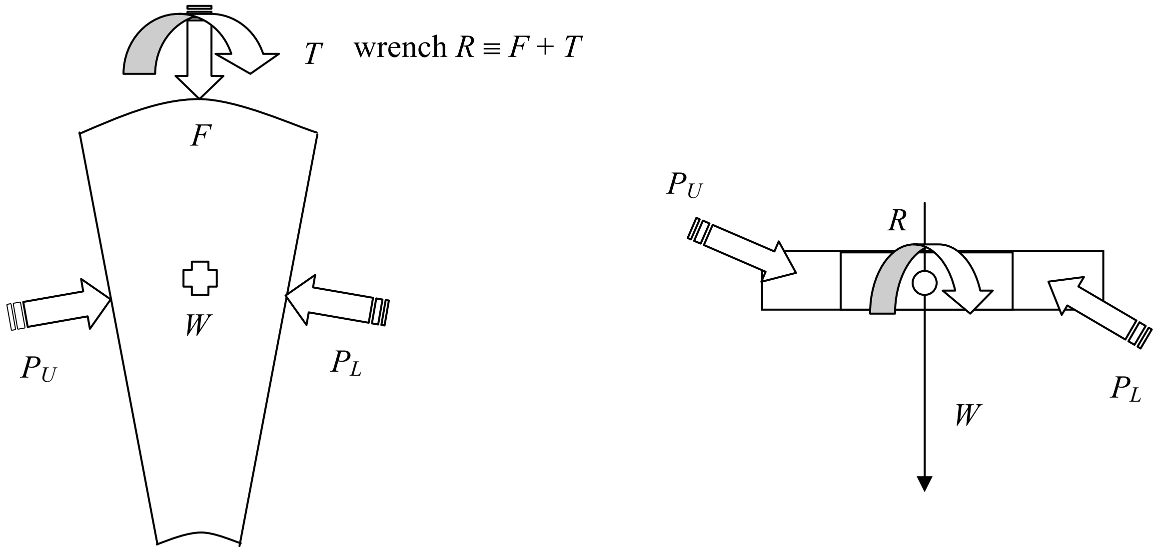


FIGURE 8. Force system for a curved arch.

6. CLOSING REMARKS

The decision to formulate the limit analyses in an Excel spreadsheet was influenced by its convenience for small problems and the availability of the GRG non-linear method – albeit as a “black box” algorithm. The results obtained prompt further investigation of this method for much larger problems using either programming in Visual Basic within Excel or using languages such as Fortran90. Although the GRG algorithm suffers from questions regarding the local or global nature of a solution, each solution has the advantage of being “safe” if the lower bound theorem of plasticity can be invoked.

REFERENCES

[1] Price, S, Cantilevered staircases, *Architectural Research Quarterly*, 1, *construction*, **1996**, 76-87.
[2] Price, S, and Rogers, H, Stone cantilevered staircases, *The Structural Engineer*, 83(2), **2005**, 29-36.
[3] Parker, D, Fears for cantilever stairs after collapse, *NCE*, 25 March, **2004**, 10
[4] Maunder, EAW, Limit analysis of masonry structures based on discrete elements, in CA Brebbia & RJB Frewer, eds, *Structural Repair and Maintenance of Historic Buildings III*, Computational Mechanics Publications, **1993**, 367-374.
[5] Maunder, EAW, Staircases as cantilevers or arches? – a question for limit analysis, in C Modena, PB Lourenco, P Roca, eds, *Structural Analysis of Historical Constructions*, A.A.Balkema Publishers, **2005**, 569-576.