

# konik: a concrete plasticity model

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The model is constructed from the peak nominal stress (PNS) surfaces that have been identified from laboratory experiments. The limiting envelope is expressed in terms of (i) a deviatoric shape function  $r$ , (ii) the compression meridian  $\bar{\rho}_{ps_c}$  and (iii) the extension meridian  $\bar{\rho}_{ps_e}$ .

$$\bar{\rho}_{ps_c} = \sqrt{c_1 \bar{\xi}^2 - c_2 \bar{\xi} + c_3} \quad \bar{\rho}_{ps_e} = \sqrt{e_1 \bar{\xi}^2 - e_2 \bar{\xi} + e_3} \quad (1)$$

$\xi = \sigma_{ii}/\sqrt{3}$  and  $\rho = \sqrt{s_{ij}s_{ij}}$  where the deviatoric stresses  $s_{ij} = \sigma_{ij} - \delta_{ij}\xi/\sqrt{3}$ . The hyperbolic form ensures that the surface never closes on the *compressive* hydrostatic axis, while intersecting the tensile hydrostatic axis normally.  $c_1$ - $c_3$ ,  $e_1$ - $e_3$  are six dimensionless positively-valued material constants. Considering the common point of tensile closure of the PNS surface on the hydrostatic axis, the number of material constants reduces to five. These may be determined from the following set of experimental data: uniaxial compression ( $f_c$ ), uniaxial tension ( $f_t$ ), equal biaxial compression ( $f_{bc}$ ), hydrostatic tension ( $f_{ht}$ ) and high level triaxial confinement on the compression meridian ( $\bar{\xi}_{tc}$ ,  $\bar{\rho}_{tc}$ ). The overbar indicates normalization of the stresses with respect to  $f_c$ .

$$c_1 = \frac{-\frac{2}{3}(\bar{\xi}_{tc} + \frac{1}{\sqrt{3}}) - (\bar{\rho}_{tc}^2 - \frac{2}{3})(\bar{\xi}_{ht} + \frac{1}{\sqrt{3}})}{(\bar{\xi}_{ht} - \frac{1}{3})(\bar{\xi}_{tc} + \frac{1}{\sqrt{3}}) - (\bar{\xi}_{tc} - \frac{1}{3})(\bar{\xi}_{ht} + \frac{1}{\sqrt{3}})} \quad (2)$$

$$c_2 = \frac{(\frac{2}{3} - \bar{\rho}_{tc}^2)(\bar{\xi}_{ht} - \frac{1}{3}) - \frac{2}{3}(\bar{\xi}_{tc} - \frac{1}{3})}{(\bar{\xi}_{ht} - \frac{1}{3})(\bar{\xi}_{tc} + \frac{1}{\sqrt{3}}) - (\bar{\xi}_{tc} - \frac{1}{3})(\bar{\xi}_{ht} + \frac{1}{\sqrt{3}})} \quad (3)$$

$$c_3 = -c_1 \bar{\xi}_{ht}^2 + c_2 \bar{\xi}_{ht} \quad (4)$$

$$e_1 = \frac{\frac{2}{3} \bar{f}_t^2 (-\frac{2}{\sqrt{3}} \bar{f}_{bc} - \bar{\xi}_{ht}) - \frac{2}{3} \bar{f}_{bc}^2 (\frac{1}{\sqrt{3}} \bar{f}_t - \bar{\xi}_{ht})}{(\frac{1}{3} \bar{f}_t^2 - \bar{\xi}_{ht}^2)(-\frac{2}{\sqrt{3}} \bar{f}_{bc} - \bar{\xi}_{ht}) - (\frac{1}{\sqrt{3}} \bar{f}_t - \bar{\xi}_{ht})(\frac{4}{3} \bar{f}_{bc}^2 - \bar{\xi}_{ht}^2)} \quad (5)$$

$$e_2 = \frac{\frac{2}{3} \bar{f}_{bc}^2 (\bar{\xi}_{ht} - \frac{1}{3} \bar{f}_t^2) - \frac{2}{3} \bar{f}_t^2 (\bar{\xi}_{ht} - \frac{4}{3} \bar{f}_{bc}^2)}{(\frac{1}{3} \bar{f}_t^2 - \bar{\xi}_{ht}^2)(-\frac{2}{\sqrt{3}} \bar{f}_{bc} - \bar{\xi}_{ht}) - (\frac{1}{\sqrt{3}} \bar{f}_t - \bar{\xi}_{ht})(\frac{4}{3} \bar{f}_{bc}^2 - \bar{\xi}_{ht}^2)} \quad (6)$$

$$e_3 = -e_1 \bar{\xi}_{ht}^2 + e_2 \bar{\xi}_{ht} \quad (7)$$

where  $\bar{\xi}_{ht} = \sqrt{3} \bar{f}_{ht}$ .  $c_1$  and  $e_1$  can be thought of as measures of internal friction since  $\sqrt{c_1}$  and  $\sqrt{e_1}$  are the slopes ( $\partial \bar{\rho}_{ps} / \partial \bar{\xi}$ ) of the compression and extension asymptotes. Thus, as  $\bar{\xi}$  tends to  $-\infty$ , so the ratio  $\bar{\rho}_{ps_e} / \bar{\rho}_{ps_c}$  tends to  $\sqrt{e_1 / c_1}$ . Decreasing either  $c_2$  or  $e_2$  has the effect of decreasing the hydrostatic tensile strength while increasing the triaxial compression and extension strengths.  $c_3$  and  $e_3$  represent measures of cohesion (when  $\bar{\xi} = 0$ ,  $\bar{\rho}_{ps_c} = \sqrt{c_3}$  and  $\bar{\rho}_{ps_e} = \sqrt{e_3}$ ). The deviatoric shape function  $r(\varphi)$  follows the elliptic form first presented in [1] where  $\varphi = \frac{1}{3} \arcsin(-3J_3\sqrt{3}/(2J_2^{3/2}))$ ,  $J_2 = \frac{1}{2} s_{ij}s_{ji}$ ,  $J_3 = \frac{1}{3} s_{ij}s_{jk}s_{ki}$  and  $r$  expresses the ratio  $\rho_{ps} / \rho_{ps_c}$ . The function creates a smooth locus with 6-fold symmetry in the deviatoric planes that remains convex when  $r(-\frac{\pi}{6}) \geq \frac{1}{2}$ .

$$r = \frac{r_1 \alpha + \sqrt{2r_1 \alpha^2 + r_2}}{2r_1 \alpha^2 + 1} \quad (8)$$

$\alpha = \cos(\varphi + \frac{\pi}{6})$ ,  $r_0 = \frac{\rho_{ps_e}}{\rho_{ps_c}}$ ,  $r_1 = 2(1 - r_0^2)/(2r_0 - 1)^2$  and  $r_2 = r_0(5r_0 - 4)/(2r_0 - 1)^2$ . The hardening yield function has been constructed to provide a smoothly transforming surface which is divided into compactive and dilative regions. The former is developed from a rotated ellipse which closes on the hydrostatic axis in the compression region. The first stage in the

construction of the hardening yield surface involves definition of the plastic volumetric transition (PVT) locus  $\bar{\rho}_{vt_c} = c_4 \bar{\rho}_{ps_c}$ .

Consider a point  $\bar{\xi}_{hc}$  on the hydrostatic axis, in the  $\bar{\xi}$ - $\bar{\rho}_c$  plane. A line with slope  $c_5$  connects  $\bar{\xi}_{hc}$  to an intersection point on the PVT locus at  $\bar{\xi}_{vt}$ ,  $\bar{\rho}_{vt_c}$ . In order to arrive at the expressions that define the rotated ellipse, we make use of local axes  $x_1$  and  $y_1$ , although finally we will express the surface in terms of the invariants  $\bar{\xi}$  and  $\bar{\rho}$ . Let the center of an ellipse be located at the origin of the  $x_1$ ,  $y_1$  coordinates. The major axis coincides with the  $x_1$  axis and the minor axis coincides with the  $y_1$  axis. We write the equations for the ellipse in bipolar form

$$\begin{cases} x_1 = a \cos(\theta) \\ y_1 = b \sin(\theta) = a c_6 \sin(\theta) \end{cases} \quad (9)$$

where  $\theta$  is the counter-clockwise angle measured from the  $x_1$  axis to a point on the circle used in the construction of the bipolar expressions.  $c_6$  identifies the aspect ratio ( $b/a$ ) where  $a$  and  $b$  are the ellipse major and minor semi-axis lengths. Consider now another rectangular coordinate system  $x_2$ ,  $y_2$  formed by rotating clockwise  $\theta_{12}$  (measured from the  $x_1$  axis) about the origin. Then we make the translation from  $x_2$ ,  $y_2$  to the  $\bar{\xi}$ ,  $\bar{\rho}$  coordinate system such that

$$\begin{cases} \bar{\xi} = a \cos(\theta) \cos(\theta_{12}) - a c_6 \sin(\theta) \sin(\theta_{12}) + \bar{\xi}_0 \\ \bar{\rho}_c = a \cos(\theta) \sin(\theta_{12}) + a c_6 \sin(\theta) \cos(\theta_{12}) + \bar{\rho}_0 \end{cases} \quad (10)$$

where  $\bar{\xi}_0$  and  $\bar{\rho}_0$  are dependent upon  $c_4$ ,  $c_5$  and  $c_6$ . Defining  $\alpha_0 = \bar{\rho}_c / \bar{\rho}_{ps_c}$ , we now determine the set of intermediate variables  $a$ ,  $\theta_{12}$ ,  $\bar{\xi}_0$  and  $\bar{\rho}_0$  from the material constants  $c_4$ ,  $c_5$  and  $c_6$  and consideration of the constraints acting on the ellipse. These constraints (i) force the ellipse to pass through  $\bar{\xi}_{vt}$ ,  $\bar{\rho}_{vt}$  with a tangent parallel to the hydrostatic axis and (ii) pass through  $\bar{\xi}_{hc}$  with a tangent normal to the hydrostatic axis.

$$\bar{\rho}_c |_{\theta_{hc}} = 0 \quad \bar{\rho}_c |_{\theta_{vt}} = \bar{\rho}_{vt} \quad \left. \frac{\partial \bar{\xi}}{\partial \theta} \right|_{\theta_{hc}} = 0 \quad \left. \frac{\partial \bar{\rho}_c}{\partial \theta} \right|_{\theta_{vt}} = 0 \quad (11)$$

where  $\theta_{hc}$  and  $\theta_{vt}$  are those angles associated with the hydrostatic intersection point and the volumetric transition point respectively. Differentiating (10)<sub>1</sub> and substituting into (11)<sub>3</sub>, after rearrangement gives

$$\theta_{hc} = \pi + \arctan(-c_6 \tan \theta_{12}) \quad \dots \text{ similarly} \quad \theta_{vt} = \arctan(c_6 \cot \theta_{12}) \quad (12)$$

Considering (10) at locations  $\bar{\xi}_{hc}$  and  $\bar{\rho}_{hc}$ , we obtain

$$\begin{cases} \bar{\xi}_0 = -a \cos(\theta_{hc}) \cos(\theta_{12}) + a c_6 \sin(\theta_{hc}) \sin(\theta_{12}) + \bar{\xi}_{hc} \\ \bar{\rho}_0 = -a \cos(\theta_{hc}) \sin(\theta_{12}) - a c_6 \sin(\theta_{hc}) \cos(\theta_{12}) \end{cases} \quad (13)$$

The intermediate variables  $a$  and  $\theta_{12}$  are defined next. Considering the volumetric transition point and using (10)<sub>1</sub>, we have

$$\bar{\xi}_{vt} = a \cos(\theta_{vt}) \cos(\theta_{12}) - a c_6 \sin(\theta_{vt}) \sin(\theta_{12}) + \bar{\xi}_0 \quad (14)$$

Substituting (13)<sub>1</sub> into (14) and rearranging gives

$$a = \frac{\bar{\xi}_{vt} - \bar{\xi}_{hc}}{(\cos(\theta_{vt}) - \cos(\theta_{hc})) \cos(\theta_{12}) - c_6 (\sin(\theta_{vt}) - \sin(\theta_{hc})) \sin(\theta_{12})} \quad (15)$$

Substituting (10)<sub>2</sub> into (11)<sub>4</sub> gives

$$\bar{\rho}_{vt} = a (\cos(\theta_{vt}) \sin(\theta_{12})) + a c_6 (\sin(\theta_{vt}) \cos(\theta_{12})) + \bar{\rho}_0 \quad (16)$$

Substituting (13)<sub>2</sub> into (16), solving for  $a$  and substituting into  $\bar{\rho}_{vt} = c_5 (\bar{\xi}_{vt} - \bar{\xi}_{hc})$  gives an expression for  $a$  which may be equated with (15), allowing  $a$  to be eliminated.

$$-c_6 \frac{(\cos(\theta_{vt}) - \cos(\theta_{hc})) (c_5 \cos(\theta_{12}) - \sin(\theta_{12}))}{(\sin(\theta_{vt}) - \sin(\theta_{hc})) (c_5 \sin(\theta_{12}) + \cos(\theta_{12}))} = 0 \quad (17)$$

It is not possible to arrive at an explicit expression for  $\theta_{12}$ , thus (17) has to be solved iteratively, for example using the Newton-Raphson scheme. We now derive an explicit expression for  $\alpha_0$  in terms of  $\bar{\xi}$ . Subtracting  $\bar{\xi}_0$  from (10)<sub>1</sub> and multiplying by  $\cos(\theta_{12})$  and  $\sin(\theta_{12})$ , then performing

a similar operation on  $(10)_2$  and summing and finding the difference of the resultant expressions, gives two terms which are divided by  $a$  and  $a c_6$  respectively. Squaring these and adding provides

$$+ \frac{1}{(c_6)^2} \left( \cos(\theta_{12})(\bar{\xi} - \bar{\xi}_0) + \sin(\theta_{12})(\bar{\rho}_c - \bar{\rho}_0) \right)^2 + \frac{1}{(c_6)^2} \left( \cos(\theta_{12})(\bar{\rho}_c - \bar{\rho}_0) - \sin(\theta_{12})(\bar{\xi} - \bar{\xi}_0) \right)^2 = a^2 \quad (18)$$

We define  $\alpha_1$  and  $\alpha_2$  as

$$\alpha_1 = (\bar{\xi} - \bar{\xi}_0) \cos(\theta_{12}) - \bar{\rho}_0 \sin(\theta_{12}) \quad \alpha_2 = -(\bar{\xi} - \bar{\xi}_0) \sin(\theta_{12}) - \bar{\rho}_0 \cos(\theta_{12}) \quad (19)$$

Substituting these into (18) and rearranging gives

$$\alpha_3 \bar{\rho}_c^2 + \alpha_4 \bar{\rho}_c + \alpha_5 = 0 \quad (20)$$

where

$$\alpha_3 = (\sin(\theta_{12}))^2 + \frac{(\cos(\theta_{12}))^2}{c_6^2} \quad \alpha_4 = 2 \alpha_1 \sin(\theta_{12}) + \frac{2}{c_6^2} \alpha_2 \cos(\theta_{12}) \quad \alpha_5 = \frac{\alpha_2^2}{c_6^2} + \alpha_1^2 - a^2 \quad (21)$$

Solving (20) for  $\bar{\rho}_c$  provides the expression we seek for the compactive region

$$\bar{\rho}_c = \frac{1}{2 \alpha_3} \left( -\alpha_4 + \sqrt{\alpha_4^2 - 4 \alpha_3 \alpha_5} \right) \quad \bar{\xi} \leq \bar{\xi}_{vt} \quad (22)$$

The dilative region of the hardening surface is constructed from enforcing the following constraints at  $\bar{\xi}_{vt}$  and  $\bar{\xi}_{ht}$ : (i) the dilative region exhibits  $C_2$  continuity with the compactive region at the PVT point, (ii) the deviatoric hardening measure,  $\alpha_0$ , equals one at  $\bar{\xi}_{ht}$  and (iii) the tangent to the  $\alpha_0$  versus  $\bar{\xi}$  curve is zero at  $\bar{\xi}_{ht}$ . Given

$$\left. \frac{\partial^2 \bar{\rho}_c}{\partial \bar{\xi}^2} \right|_{\bar{\xi}_{vt}} = - \frac{\cos(\theta_{vt}) \sin(\theta_{12}) + c_6 \sin(\theta_{vt}) \cos(\theta_{12})}{a(\sin(\theta_{vt}) \cos(\theta_{12}) + c_6 \cos(\theta_{vt}) \sin(\theta_{12}))^2} = \alpha_6 \quad (23)$$

and expressing the constraints in terms of  $\alpha_0$ , we have

$$\alpha_0|_{\bar{\xi}_{vt}} = c_4 \quad \left. \frac{\partial \alpha_0}{\partial \bar{\xi}} \right|_{\bar{\xi}_{vt}} = - \frac{c_4 \alpha_7}{\sqrt{c_1 \bar{\xi}_{vt}^2 - c_2 \bar{\xi}_{vt} + c_3}} = \alpha_9 \quad \left. \frac{\partial^2 \alpha_0}{\partial \bar{\xi}^2} \right|_{\bar{\xi}_{vt}} = \frac{\alpha_6 - 2 \alpha_9 \alpha_7 - c_4 \alpha_8}{\sqrt{c_1 \bar{\xi}_{vt}^2 - c_2 \bar{\xi}_{vt} + c_3}} = \alpha_{10} \quad (24)$$

where

$$\alpha_7 = \frac{2c_1 \bar{\xi}_{vt} - c_2}{2\sqrt{c_1 \bar{\xi}_{vt}^2 - c_2 \bar{\xi}_{vt} + c_3}} \quad \alpha_8 = \frac{4c_1(c_1 \bar{\xi}_{vt}^2 - c_2 \bar{\xi}_{vt} + c_3) - (2c_1 \bar{\xi}_{vt} - c_2)^2}{4(c_1 \bar{\xi}_{vt}^2 - c_2 \bar{\xi}_{vt} + c_3)^{3/2}} \quad (25)$$

At  $\bar{\xi}_{ht}$  two constraints exist

$$\alpha_0|_{\bar{\xi}_{ht}} = 1 \quad \text{and} \quad \left. \frac{\partial \alpha_0}{\partial \bar{\xi}} \right|_{\bar{\xi}_{ht}} = 0 \quad (26)$$

We now construct a surface in the dilative region by use of the following conic form (where  $\alpha_{11-15}$  are fixed for a given surface, as is  $\bar{\xi}_{vt}$ )

$$\alpha_{11}(\bar{\xi} - \bar{\xi}_{vt})^2 + \alpha_{12}(\bar{\xi} - \bar{\xi}_{vt})(\alpha_0 - c_4) + (\alpha_0 - c_4)^2 + \alpha_{13}(\bar{\xi} - \bar{\xi}_{vt}) + \alpha_{14}(\alpha_0 - c_4) + \alpha_{15} = 0 \quad (27)$$

Substituting (24<sub>1</sub>) into (27), we find that  $\alpha_5 = 0$ . Inserting this into (27) and rearranging, differentiating twice (with respect to  $\bar{\xi}$ ) and substituting the constraint conditions (24<sub>2</sub>) and (24<sub>3</sub>), then inserting (26)<sub>1</sub> into (27) and (26)<sub>2</sub> into the derivative of (27) gives

$$\begin{aligned} \alpha_{13} + \alpha_{14} \alpha_9 &= 0 \\ 2\alpha_{11} + 2\alpha_{12} \alpha_9 + 2\alpha_9^2 + \alpha_{14} \alpha_{10} &= 0 \\ \alpha_{11}(\bar{\xi}_{ht} - \bar{\xi}_{vt})^2 + \alpha_{12}(\bar{\xi}_{ht} - \bar{\xi}_{vt})(1 - c_4) & \\ + (1 - c_4)^2 + \alpha_{13}(\bar{\xi}_{ht} - \bar{\xi}_{vt}) + \alpha_{14}(1 - c_4) &= 0 \\ 2\alpha_{11}(\bar{\xi}_{ht} - \bar{\xi}_{vt}) + \alpha_{12}(1 - c_4) + \alpha_{13} &= 0 \end{aligned} \quad (28)$$

Eliminating  $\alpha_{12}$  by manipulating (28)<sub>3</sub> and (28)<sub>4</sub>, then solving for  $\alpha_{11}$  gives

$$\alpha_{11} = \frac{1 - c_4}{(\bar{\xi}_{ht} - \bar{\xi}_{vt})^2} (1 + \alpha_{14} - c_4) \quad (29)$$

Solving (28)<sub>1</sub> gives  $\alpha_{13} = -\alpha_{14}\alpha_9$  Substituting (29) and  $\alpha_{13}$  into (28)<sub>4</sub>, and solving for  $\alpha_{12}$

$$\alpha_{12} = \frac{1}{1 - c_4} \left( \alpha_{14}\alpha_9 - \frac{2(1 - c_4)}{\bar{\xi}_{ht} - \bar{\xi}_{vt}} (1 + \alpha_{14} - c_4) \right) \quad (30)$$

Substituting (30) and (29) into (28)<sub>2</sub>, and solving for  $\alpha_{14}$

$$\alpha_{14} = -\frac{2(1 - c_4)^2 - 4\alpha_9(1 - c_4)(\bar{\xi}_{ht} - \bar{\xi}_{vt}) + 2\alpha_9^2(\bar{\xi}_{ht} - \bar{\xi}_{vt})^2}{2(1 - c_4) - 4\alpha_9(\bar{\xi}_{ht} - \bar{\xi}_{vt}) + \left(\frac{2\alpha_9^2}{1 - c_4} + \alpha_{10}\right)(\bar{\xi}_{ht} - \bar{\xi}_{vt})^2} \quad (31)$$

We now consider (27) as a quadratic in  $\alpha_0$  which leads to the solution

$$\bar{\rho}_c = \bar{\rho}_{ps_c} \left( c_4 + \frac{1}{2}(-\alpha_{16} \pm \alpha_{18}) \right) \quad \bar{\xi} > \bar{\xi}_{vt} \quad (32)$$

Whether  $\alpha_{18}$  is added or subtracted in (32) depends on the sign of  $\alpha_{19}$  ( $+\alpha_{18}$  when  $\alpha_{19} < 0$ ,  $-\alpha_{18}$  when  $\alpha_{19} \geq 0$ ).

$$\begin{aligned} \alpha_{16} &= \alpha_{12}(\bar{\xi} - \bar{\xi}_{vt}) + \alpha_{14} & \alpha_{17} &= \alpha_{11}(\bar{\xi} - \bar{\xi}_{vt})^2 + \alpha_{13}(\bar{\xi} - \bar{\xi}_{vt}) + \alpha_{15} \\ \alpha_{18} &= \sqrt{\alpha_{16}^2 - 4\alpha_{17}} & \alpha_{19} &= \alpha_{12}^2 - 4\alpha_{11} \end{aligned} \quad (33)$$

The degree of hardening experienced by the material is characterized by  $\bar{\xi}_{hc}$ .

$$\bar{\xi}_{hc} = \frac{c_7 \alpha_{20}}{(\alpha_{20} - 1) \sum_{m=0}^3 (\alpha_{20})^m} \quad \text{where} \quad \alpha_{20} = c_8 + (1 - c_8)k_h \quad (34)$$

$c_8$  takes on the role of identifying the size of the initial elastic nucleus (that is, the initial yield surface). The internal variable  $k_h$  ( $\int dk_h$ ) is tied to the effective plastic strain via the following rate relationship

$$\dot{k}_h = \frac{\sqrt{\frac{2}{3}\dot{\epsilon}_{ij}^p \dot{\epsilon}_{ji}^p}}{\alpha_{21}} \quad \alpha_{21} = c_9(\bar{\xi}_{ht} - \bar{\xi})(1 - \tanh(c_{10}\bar{\xi} + c_{11})) (k_h)^{c_{12}} (c_{13}(\bar{\rho})^{c_{14}} + 1) \quad (35)$$

To replicate softening, the PNS surface translates along the hydrostatic axis (in the compressive direction) such that, in the residual state, all hydrostatic tensile strength disappears. Motion of the PNS surface is achieved through introduction of the softening variable  $k_s$  modifying (1)

$$\bar{\rho}_c = \sqrt{c_1(\bar{\xi} - \bar{\xi}_{ht}(k_s - 1))^2 - c_2(\bar{\xi} - \bar{\xi}_{ht}(k_s - 1)) + c_3} \quad (36)$$

$$\bar{\rho}_e = \sqrt{e_1(\bar{\xi} - \bar{\xi}_{ht}(k_s - 1))^2 - e_2(\bar{\xi} - \bar{\xi}_{ht}(k_s - 1)) + e_3} \quad (37)$$

The measure of de-cohesion (reducing from 1 to 0) is defined by

$$k_s = \exp \left( -c_{15} \left( \frac{\alpha_{22}}{\alpha_{23}} \sqrt{\langle \dot{\epsilon}_i^p \rangle \langle \dot{\epsilon}_i^p \rangle} \right)^{c_{16}} \right) \quad (38)$$

$\alpha_{22}$  identifies the *characteristic length* of the *finite element*.  $\langle \dot{\epsilon}_i^p \rangle$  selects only the positive (tensile) part of the principal plastic strain rates.  $c_{17}$ , represents the relative measure of the effective fracture energy expended during uniaxial compression with respect to that consumed under uniaxial tension.  $\eta = \rho/\xi$ .

$$\alpha_{23} = \sqrt{2} \left( \frac{1}{2} - \frac{1}{\eta} \right) (1 - \cos(\varphi + \frac{\pi}{6})) (c_{17} - 1) + 1 \quad \frac{1}{\eta} < \frac{1}{\sqrt{2}} \quad \alpha_{23} = 1 \quad \frac{1}{\eta} \geq \frac{1}{\sqrt{2}} \quad (39)$$

## REFERENCES

- [1] Willam, K J and Warnke, E P, 1975, Constitutive model for the triaxial behaviour of concrete, in IABSE seminar on Concrete Structures subjected to Triaxial Stresses, Report 19, III-1, 1-30.