13^{th} ACME CONFERENCE: UNIVERSITY *of* SHEFFIELD MARCH 21-22 2005

A six-node curved membrane finite element with Loof curvatures

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Abstract

Two finite element methods for the analysis of membrane structures are presented in this paper - a 3-node constant strain trianglar element(CST) with large strains and 6-node linear strain triangle elements(LST) with large deformation. For the CST element, the cable analogy method linked with the dynamic relaxation. The significant error caused by ignoring second order('large') strains in membrane structure analysis will be demonstrated mathematically and numerically through a proposed enhanced nonlinear cable analogy formulation.Case studies comparing these two CST formulations will conclude this section of the paper.

In LST approach,a new curved 6-node triangular element coupled with the Newton-Raphson algorithm will be introduced with Loof curvatures. The introduction of curvatures in the numerical representations of membrane surfaces are demonstrated to be important terms which affect the relationship between the membrane strains and nodal displacements and in some special cases,the out-of-balance forces. Having summarised the formulation,numerical examples will be given in both the shape finding and load analysis procedures.

Finally, results of the numerical case studies analysed using the CST and LST will be compared and discussed to domonstrate the accuracies and limitations of these two finite element methodologies.

Introduction

The FEM provides the most versatile approach for the analysis of tensioned structures. 3-node triangular element with constant stress is a simplest one and popularly used in industry ,and therefore different finite element methods with higher order [1,2] have been investigated to take more accurate membrane structure analysis with less computing cost.

Membrane structures are regarded as geometrically nonlinear structures. Small displacement theory is no longer applicable to geometrically nonlinear structures[3,4] In the linear analysis, a linear relation is assumed between strains and displacements.However if there are large displacements and rotations, the nonlinear relation between strains and displacements cannot be ignored.Also, the equilibrium equation of internal and external forces should be considered in the deformed configuration.The geometrically nonlinear analysis may be described by using the total or updated lagrangian formulations. The total lagrangian formulation is derived with respect to the initial configuration.

Even with geometrically nonlinear formulation, Tabrrok and Qin[5] believe that owing to the greater geometric nonlinearity of membrane structures, it is preferable to use a dense mesh of primitive elements rather than a coarse mesh made up of higher order element.Because those classical 2-D finite elements have little resistance to the force vector normal to the element surface.The force vectors normal to one element surface are mainly carried by the adjacent elements.When large displacements normal to the membrane surface happen, the primitive elements like 3-node constant strain triangle could have better performance than those high order element by using same D.O.F.

Yexiao bing[6] has developed 2-D finite element with high order by introducing fabric curvature into finite formulation.He considers that element strains in the fabric not only change with displacements along the local element surface, but also with local element curvatures and displacements normal to the element surface.

Two finite element methods for the analysis of membrane structures are presented in this paper.A 3-node constant strain triangle element(CST) with large strains and 6-node linear strain triangle(LST) with large deformations.

Finite Element Model

3-node constant strain triangle element with large strains

3-node CST element with cable analogy is most popularly used in membrane design, because it is more straightforward for membrane structure analysis, if the membrane could be simulated to the cable net, then the F.E.M formulation applied in cable element are usable for membrane analysis. Under the assumption of small strains, the cable element is:

$$\varepsilon_i = \varepsilon_X \; cos^2\theta_i + \varepsilon_Y \; sin^2\theta_i + \gamma_{XY} sin\theta_i cos\theta_i \tag{1}$$

in which, ε_i is the i^{th} pseudo cable strain. ε_X,ε_Y,γ_{XY} are local strains in membrane element. θ is angle between pseudo cable and local X axis

But when large strain happened, nonlinear part in the simulation from membrane to cable cannot be ignored, the nonlinear formulation is:

$$\varepsilon_i = \varepsilon_x cos^2\theta_i + \varepsilon_y cos^2\theta_i + \varepsilon_{xy} sin\theta_i cos\theta_i + \Delta \tag{2}$$

in which

$$\Delta = [\frac{1}{8}(\varepsilon_x - \varepsilon_y)^2 - \frac{1}{8}\gamma_{xy}^2] \cdot sin^2 2\theta_i + \frac{1}{2}\gamma_{xy} \cdot (\varepsilon_y \cdot cos^2\theta_i + \varepsilon_x \cdot sin^2\theta_i + \varepsilon_x \cdot \varepsilon_y) \cdot sin2\theta_i$$

here,Δ is nonlinear part of the simulating formulation.

So the modified nonlinear F.E formulation have been developed for large strain:

$$[\varepsilon_p] = [B_{NL}] \; \{\varepsilon_i\}, \qquad K = \int \; [B_{NL}]^T \; [E] \; [B_{NL}] \cdot dV \tag{3}$$

in which, ε_p are local principle strains, $[\varepsilon_i]$ are pseudo cable strains, $[E]$ is elastic modulus matrix, $[B_{NL}]$ is nonlinear cable analogy formulation.

$$[B_{NL}] = \begin{bmatrix} 0 & 0 & l_3^{-1} \\ \frac{a_2(2+\varepsilon_1)l_1^{-1}}{A(2+\alpha)} & \frac{b_2(2+\varepsilon_2)l_2^{-1}}{A(2+\alpha)} & \frac{c_2(2+\varepsilon_3)l_3^{-1}}{A(2+\alpha)} \\ \frac{a_3(2+\varepsilon_1)l_1^{-1}}{2(1+\varepsilon_3)B} & \frac{b_3(2+\varepsilon_2)l_2^{-1}}{2(1+\varepsilon_3)B} & \frac{c_3(2+\varepsilon_3)l_3^{-1}}{2(1+\varepsilon_3)B} \end{bmatrix} \tag{4}$$

in which a_i, b_i, c_i and A are constants, α and B are terms of pseudo cable strains

6-node linear strain triangular element with large strains

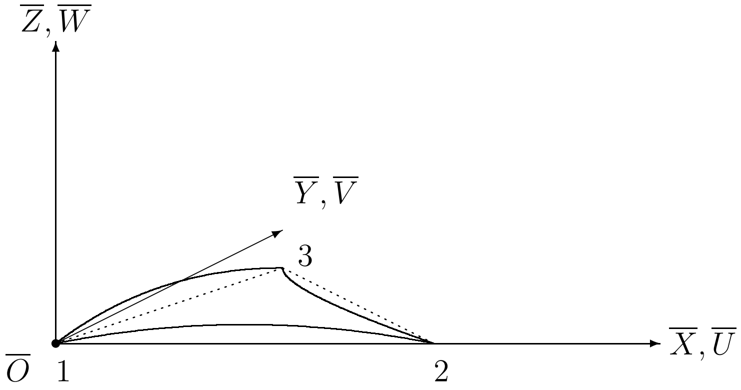

Figure 1: Local Plane Co-ordinate System

The geometry of the 6-node triangular element can be presented by $\overline{Z} = f(\overline{X}, \overline{Y})$

Then the curvature in $\overline{X}$ and $\overline{Y}$ axis:

$$K_{\overline{X}} = -\frac{\partial^2 Z}{\partial X^2}, K_{\overline{Y}} = -\frac{\partial^2 Z}{\partial Y^2} \tag{5}$$

and the torsion between $\overline{X}$ and $\overline{Y}$ axis:

$$K_{\overline{XY}} = -\frac{\partial^2 \overline{Z}}{\partial \overline{X} \partial \overline{Y}} \tag{6}$$

Nodal coordinates in Local curved coordinate system could be obtained by the formulation in previous section,area coordinate system(ξ_1,ξ_2,ξ_3) will be applied in the 6 node curved triangular element,the shape function are:

$$N_1 = \xi_1(2\xi_1 - 1), N_2 = \xi_2(2\xi_2 - 1), N_3 = \xi_3(2\xi_3 - 1)$$
$$N_4 = 4\xi_1\xi_2, N_5 = 4\xi_2\xi_3, N_6 = 4\xi_3\xi_1$$

Because the small-deflection theory of linear elasticity is inapplicable and the quadratic terms in displacement-strain relations must be taken into account.The nonlinear displacement-strain relations may be expressed as:

$$\varepsilon_X = \frac{\partial U}{\partial X} + K_{\overline{X}} W + \frac{1}{2}\left[\left(\frac{\partial U}{\partial X}\right)^2 + \left(\frac{\partial V}{\partial X}\right)^2 + \left(\frac{\partial W}{\partial X}\right)^2\right] \tag{7}$$

$$\varepsilon_Y = \frac{\partial V}{\partial Y} + K_{\overline{Y}} W + \frac{1}{2}\left[\left(\frac{\partial U}{\partial Y}\right)^2 + \left(\frac{\partial V}{\partial Y}\right)^2 + \left(\frac{\partial W}{\partial Y}\right)^2\right] \tag{8}$$

$$\gamma_{XY} = \left[\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X} + 2K_{\overline{XY}} W + \left(\frac{\partial U}{\partial X}\right)\left(\frac{\partial U}{\partial Y}\right) + \left(\frac{\partial V}{\partial X}\right)\left(\frac{\partial V}{\partial Y}\right) + \left(\frac{\partial W}{\partial X}\right)\left(\frac{\partial W}{\partial Y}\right)\right] \tag{9}$$

the equilibrium equation could be written as:

$$F = (K_E + K_\delta)d + f_e$$

in which,F is load Force, $K_E$is elastic stiffness matrix, K_δ is geometric stiffness matrix and f_e is out-of-balance vector.

Numerical Examples

Comparison between linear and nonlinear cable analogy formulation

Using a patch test to compare the results from linear and nonlinear cable analogy formulation with different strain deformations.

Form-finding

Two numerical examples have been given for form-finding procedure by 6-node curved finite element module: catenoid with uniform stresses and membrane plate tensioned by boundary cables.

Loading Analysis

A uniform Load is carried on the tensioned membrane plate with tensioned boundary cables.

Conclusion

From the comparison between linear and nonlinear cable analogy formulation for 3 node triangle element,it is apparently that accuracies of the two F.E.M decrease with increasing membrane strain.In small strain field, both of them have same good performance,but nonlinear formulation has obvious improvement over linear formulation under large strains.

For LST element model,Loof curvatures enable the 2-D plate element model have the stiffness in the direction normal to the local element surface,the magitude of the stiffness deals with loof curvatures and material properties.The applied force vectors normal to the element surface are not only carried by adjacent elements but also by the element itself.When a membrane with curved shape is under tension,the out-of-balance force normal to the membrane surface is the sum of $Curvature \times Stress - Force$ in two principal directions.In the Shape-finding procedure, the inital stresses are constant and Forces equal to zero, so the sum of curvatures of each element in two principal directions equal to zero,which is defined "'Minimal surface"'. Compared with results from CST finite element model,according to the results with same accuracy by using same D.O.F, LST finite element model could give detailed stress and strain distribution,which is very important for wrinking procedure in membrane structure analysis.

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